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MEDICAL STATISTICS FROM GRAUNT TO FARR

(Concluded*)

BY MAJOR GREENWOOD

VI. SOME ENGLISH MEDICAL STATISTICIANS IN THE EIGHTEENTH CENTURY

AFTER Petty more than fifty years passed before another Fellow of the College of Physicians took an interest in statistics, and he, if less eminent in political arithmetic, was much more eminent in the art of medicine; he was the elder Heberden. Heberden published no statistical work over his name, but there seems no reason to doubt the accuracy of his son's statement that the quarto volume containing a collection of the yearly Bills of Mortality in London from 1657 to 1758 and various essays was financed by Dr William Heberden and that he wrote the preface. This preface, an essay of 15 pages which ends rather abruptly, could hardly have been written by a layman. The following passage illustrates my remark:

> The deaths imputed to the measles are very remarkably different in different years; and yet it is possible that this disease is not in reality so very irregularly epidemical or fatal, as by the bills it appears to be. The scarlet fever and malignant sore throat often occasion such appearances upon the skin, as may easily be mistaken for the measles by better judges than the mothers and nurses, who thinking themselves able to distinguish this distemper, and equal to the management of it, often call in no other assistance. This mistake is well known to have been sometimes made within these few years, during which the scarlet fever and malignant sore throat have been so common. It may perhaps have happened in every year, in which an extraordinary number of deaths are charged to the measles: and consequently those two formidable distempers, (if they are two distinct distempers, and not one and the same) being disguised under the name of the measles, may have been older, and more general than is usually imagined.

The writer's observations upon the disappearance of plague have also something of a professional air—the fact that they are decidedly confused is no argument to the contrary. Sydenham taught (*Obs. Med.* **2**, 2) that plague depended upon (*a*) a special disposition of the atmosphere, (*b*) the transmission of an infecting matter, and held (*a*) to be primary, i.e. that without the atmospheric constitution there would be no epidemic. Heberden thinks the decline of the plague was due to the rebuilding of the city and—'probably the most effective'—the great quantity of water from the Thames and the New River, 'which, for the last century, has washed the houses so plentifully, and afterwards running down into the kennels and common sewers, constantly hinders, or weakens the tendency to putrefaction'. Heberden, unlike Sydenham, who

* The earlier sections were printed in *Biometrika*, **32**, 101–27 and 203–25.

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believed the secret of the atmospheric constitution beyond the wit of man, seems to have attributed it to 'putrefaction', but, like Sydenham, attributed more importance to the atmospheric than the infective factor.

For the rest, Heberden continued Graunt's criticism of the material. In particular he gives good reason for thinking that beyond the omission of dissenters' christenings and burials, an important error arises from a balance of outward burials, that more coffins are taken out of London to be buried than are brought in. From an enumeration in Westminster he concludes that the deaths within the Bills are 20 % too few. He comments on an apparent increase in certain forms of death, such as apoplexies, lethargies and palsies. 'The practice of drinking spirituous liquors must, probably, answer for some part of this: and it might be of public use, if some attention were paid to the finding out of the other causes.' Rather optimistically, he thinks abundant amends might be made for these increases by the control of smallpox through inoculation. Upon this he makes a comment which has a very modern ring. 'For while inoculation prevails only among a part of any number of people, who all have an intercourse with one another, it may occasion as many deaths by spreading the distemper in, as it is called, the natural way, as it prevents among those, on whom it is practised.'

The volume contains a reprint of Graunt's work, of one of Petty's papers, and a new essay by Corbyn Morris. This essay, which shows signs of improving statistical technique, is not of medical interest except in its collection of deaths in age groups—an operation rendered possible by the introduction, in 1727, of an age classification of all deaths. Twelve age groups were given. In Heberden's preface the importance of classifying by cause of death and age is emphasized.

From this material a life table was calculated by a fellow of the Royal Society named Postlethwaite (at the request of Heberden). This table, based upon deaths alone, is, for reasons already stated, of little value.

I doubt whether even the relative mortalities are correctly shown. The age distribution of the Bills (after 1726) for deaths was: under 2, 2–5, 5–10 and thence forward by decennia. Consequently one must distribute the deaths into single years of life upon some principle of interpolation. Neither J.P. (Postlethwaite) nor J.S. (Smart)—who made a life table for the first ten of the thirty years—states the principle on which he worked. But J.P. assigns 250 of the 363 deaths at ages under 2 to the first year of life and J.S. 290 of the 386 he had to manipulate, respectively 68·9 and 75·1 % of the total mortality under 2. Both ratios are much less than given by Halley's table, 83·3 %, which itself agrees admirably with the latest English population table (E.L. No. 10, Males) 83·5 %. If we applied the 83 % ratio it would raise the rate of infant mortality to 301 per 1000. Taking the figures simply as they stand, the survivors to age 6 years are fewer than Graunt estimated—54 % not 64 % survive.

Although, as has been said, the arithmetical values are very suspect, the

indication they give may be towards the truth. Creighton gave good reasons for concluding that London after the extinction of the Plague was less, not more, healthy. These reflexions are not without importance for they help to explain a certain fatalism, a scepticism as to the possibility of reducing the death-rate, which is noticeable in both statistical and medical literature for some time to come.

The next writer to be noticed is Thomas Short. Of this industrious investigator even Sir Norman Moore could obtain few personal particulars. He may have been born in 1690 and he died in 1772. He practised in Sheffield and was a Doctor of Medicine of a Scottish university but not a licentiate of the College. His principal works, *A General Chronological History of the Air, Weather, Seasons, Meteors, etc.*, published in 1749, and *New Observations, Natural, Moral, Civil, Political and Medical on City, Town and Country Bills of Mortality*, published in 1750, are differently assessed by the greatest historian of British medicine. Creighton pronounces the former to be rubbish but gives Short a not very hard pat on the back for the latter. 'That so much statistical or arithmetical zeal and exhaustiveness (in the work of 1750) should go with so total a deficiency of the critical and historical sense (in the work of 1749) is noteworthy, and perhaps not unparalleled in modern times' (Creighton, *Hist. of Epidem. in Britain*, **1**, 405). Creighton's not very wide intellectual charity did not embrace statisticians.

It must be admitted that Short is decidedly *not* a writer to commend himself to an orderly minded, careful scholar from Aberdeen. Had he lived a century later we might have supposed that his literary model was Mrs Nickleby—he just runs on and on. A table (which must have been most troublesome to compile) of monthly christenings and weddings in various towns (in three extending over more than 150 years) leads him from arithmetical comparisons of the months most apt for procreation to a vigorous denunciation of luxury, of polygamy, of taxing common necessities, of the sale of army commissions, of novel reading, boarding schools and much else. But, although few if any would be able to read Short straight through without a rest, a good many less entertaining books might be included in a bed-side book case.

There is scarcely anything within the range of human interests upon which Dr Short has not something to say. On the whole he took a gloomy view of modern life in general and of his faculty in particular, and remarked that the 'improvements in surgery in general, have far out-stripped those in physick'. Surgeons he found to have generally less learning than physicians, but compensated for this by a closer application to the study of their own profession 'without jumbling the finite mind, and mixing studies of a different nature from their own, as of the dramatists, poets, classics, architecture, politics, history critics, logics, etc. They are also less liable to theories and false reasonings, have not that contempt of the ancients, nor of observations built on practice, improved

and directed by the understanding, and raised to the pitch of truth by a long enquiry into the effects of diseases and medicines.'

Short began his book with a clear plan—long before he had finished it the plan became an inextricable confusion. He argued that a statistical measure of health could not be obtained from the data of towns in general and the capital in particular, partly owing to the inaccuracy of the data, partly owing to the fact that towns attracted newcomers and were not maintained by the balance of births and deaths within the community. So he collected material from country parishes (he also obtained data from towns but the country parishes were his prime object of study). His first set of data was a collection of transcripts of the registers of eighty-three parishes. About 60% of these parishes were from Yorkshire (mainly in the neighbourhood of Sheffield) and Derbyshire, but some from as far afield as Devonshire. He set them out in two periods, the first ending before the Restoration, the second coming down to the third decade of the eighteenth century. He had another set of eighty-three parishes for which the data covered only the second period. He classified his parishes in accordance with the nature of soil, altitude, exposure, whether wooded or bare, wet or dry. Sometimes his data covered more than a century, rarely less than 20 years. He gives the total number of baptisms, of burials (sexes distinguished) and marriages and works out the various ratios, the ratio of baptisms to burials being his chief tool.

From these data he draws a great many conclusions; for instance, respecting the salubrity or insalubrity of different soils and exposures. Most of these conclusions, it may be remarked, are now part of the common stock of lay and, perhaps, professional belief. But whether Short's data were adequate to sustain the conclusions is another question.

We may begin by taking purely arithmetical points into consideration, viz. whether, *assuming* that the parishes or groups of parishes were fairly comparable and *assuming* that the ratio of baptisms to burials is a fair measure of healthiness, Short had large enough figures for his purpose. For instance, two of his conclusions were that dry open sites of moderate elevation were healthier than a clay soil. I picked out of his list nine parishes of the former and five of the latter class. In his first (pre-Restoration) period, the parishes on dry, open sites had registered 4349 baptisms and 2644 burials, a ratio of 1·64. The five parishes on clay had 2875 baptisms and 1920 burials, a ratio of 1·497. So, as he said, the dry open sites give a higher ratio. But what is the order of magnitude of the error of sampling? We may safely hold that the standard error of the number of baptisms or burials is of the order of the square root of the observed number, or the ratio of standard error to number is of order $1/n^{\frac{1}{2}}$. From this we infer that the standard error of the ratio n_1/n_2 is given by n_1/n_2 times $(1/n_1 + 1/n_2 - 2r_{n_1 n_2}.1/(n_1 n_2)^{\frac{1}{2}})^{\frac{1}{2}}$. Clearly, the correlation between numerator and denominator must be large, so that the second factor lies between $(1/n_1 + 1/n_2)^{\frac{1}{2}}$

and $(1/n_1^{\frac{1}{2}} - 1/n_2^{\frac{1}{2}})$ and will be much nearer the second value. In a sample of 1000 Registration Districts I studied many years ago, the correlation of births and deaths was 0·73 %, in our particular case the standard error of the difference between the two ratios will be likely to be not much more than 20 % of its value.

From the purely arithmetical angle, I should conclude that Short was justified in holding that his ratios did really differ significantly, as we say, from site to site. But is it fair to assume that (1) the ratio of baptisms to burials is a good index of healthiness, (2) that the comparisons are in *pari materia*? These are much more difficult questions.

So far as concerns modern experience, it is clear that the ratio of births to deaths does not give a useful index of the rate of mortality. I made an experiment on Short's lines. I took out a sample of fifty Registration Districts for the decennium 1901–10, chosen in the following way. (1) No districts with more than 10,000 births or fewer than 1000 deaths were taken. (2) Those with many institutional deaths were excluded. For each the ratio of births to deaths was calculated and the following table formed:*

Ratio of births to deaths	No. of districts	Mean of standardized death-rates
2·0–	4	10·36
1·9–	6	10·88
1·8–	7	11·04
1·7–	6	11·60
1·6–	15	11·15
1·5–	7	10·72
1·4–	4	11·48
1·3–	1	13·01

It is true that the district with highest ratio has the lowest death-rate and the district with lowest ratio the highest death-rate, but in detail there is but little correspondence. Testing the same districts on the data of 30 years earlier, 1871–80, the same result appears. It would be very rash to conclude that because a district has a ratio of births to deaths above the average its standardized death-rate is below the average.

There are many reasons why a ratio of births to deaths may be a bad measure

* The districts selected were: Hambledon, Malling, Faversham, Romney Marsh, Uckfield, New Forest, Romsey, Hartley Wintry, Royston, Winslow, Witney, Oundle, St Ives, Caxton, Whittlesea, Lexden, Risbridge, Mildenhall, Bosmere, Plomesgate, Flegg, Cricklade, Melksham, Amesbury, Sturminster, Kingsbridge, Stratton, St Columb, Langport, Dursley, Ledbury, Wem, Mastley, Meriden, Shipston on Stour, Lutterworth, Spilsby, Hayfield, Garstang, Settle, Pateley Bridge, Gt Ouseburn, Saddleworth, Thorne, Pocklington, Skirlaugh, Easingwold, Bedale, Weardale, Brampton.

and Short knew this; for instance, the deaths might be increased by immigrants or decreased by emigrants. He was a true Englishman of Whig principles, and in speaking of *foreign* registers, remarks that 'where the births vastly exceed the burials, the country is either very healthy, or it is under an arbitrary government or both'. If the former, the marriages will bear a high proportion to the baptisms, few will die in infancy. If the latter, 'tho' more males are born, yet the funerals of females far exceed them; there is little industry amongst the people, because they want property; useless standing armies are kept up in time of peace, for the grandeur of the tyrant, maintaining his tyranny and the oppression of his people'.

Quite logically he argues that if the burials greatly exceed the baptisms, either the situation is unhealthy or the government is limited (i.e. on English lines). One diagnoses the former when epidemics are frequent and the proportion of deaths in youth and childhood large; 'the latter is known from the great resort of strangers, labourers, artificers, merchants, etc., increase of business, trade and riches; or there is a large body of people mixed with the society, of different manners and principles, whose baptisms are not registered with the rest'.

Short was, then, aware of the fallacies possible, but he held that in country parishes they were not important.

I am not confident that he was wrong so far as concerns his first period data; country villages from the middle of the sixteenth century to the middle of the seventeenth century. But when we come to the next 80 years, when England was beginning to be an industrial country, the assumptions are more hazardous, and the almost invariably pessimistic conclusions he draws (it is a little odd that the population pessimists did not make more use of Short) may well be fallacious. The cause of change for the worse was, in the first place, the deluge of profaneness which came with the Restoration, in the next place the increase of 'Luxury, pride, intemperance and debauchery' associated with the growth of industry and wealth, so fortunately associated with the 'happy Revolution' and stabilized by the 'seasonable Accession of the present Royal Family'. But Short charges this intemperance and debauchery particularly upon the towns, and it does seem a little unreasonable to suppose that even the happy Revolution or the Hanoverian accession much increased the opportunities of villagers to fall into the sins of luxury and pride.

Short next considers the succession of unhealthy years in villages and towns. He nowhere states what his criterion of a sickly year was. The arm-chair statistician might suppose that he would take a ratio of baptisms to burials which fell below some assigned percentage of the average; but, in default of any specific statement, I should guess that Short called a year sickly when the burials exceeded the baptisms; since his totals show always an excess of baptisms over burials, this would seem a simple rule. But he may have been more subtle.

He remarks: 'It may be a sickly or mortal year in a town or country parish and yet the christenings may exceed the buryings considerably, either because it happens to be a very fruitful year in that place (as often tho' not generally happens) or the year may be very sickly in that parish, if compared with other years, and yet healthy if compared with other places in much worse situations and air.' This suggests that a sudden rise in the number of burials would be his criterion, and I have found half a dozen cases in his table (of nearly 400 instances, viz. country registers and market-town registers, examined from 1541 to 1741) when the burials for sickly years did not exceed the christenings.

It is a pity Short was not more explicit; his work in this field was original and suggestive. He finds that in the country parishes the frequency of unhealthy years is never more than twice in 5 or 6 and often as rarely as once in 6 or 8 years, 'which is indeed as long, if not a longer interval, than commonly happens between one visitation of smallpox and measles and another, exclusive of all other diseases'. 'When sickly years return oftenest there is a less disproportion between christenings and burials than when they come seldomest.' Healthy places have 'their frequent eruptive and inflammatory diseases', unhealthy places 'their slow intermittents, remittents, putrid and erratic fevers'. 'It is true, some rare times the former places are visited with the latter diseases, but rarely except they are epidemics; nor are they of a great spread, duration or execution. The latter's places have also the former's diseases, but (eruptive fevers excepted) more mildly and rarely; for each country or situation is more liable to some diseases than others, and by traffic and commerce endemics became epidemics, as far as air and climate will allow.' Short passes to a general survey of epidemiological history which is not, I think, without value, but has, of course, long been superseded by Creighton's classical study.

The passages upon which I have commented are all from the first quarter of Short's treatise and typify, I think, his most valuable contributions to our subject. As his book proceeds, not only does his habit of improving the occasion grow upon him but, in commenting and performing arithmetical operations upon the London Bills, he does not show to great advantage. But anybody who will trouble to dip into the book is likely to make a friend. Creighton said of him that 'his abstract results or conclusions are colourless and unimpressive, as statistical results are apt to be for the average concrete mind'. This seems to me rather misleading. I dare say nobody ever burst into tears over or was thrown into paroxysms of mirth by a statistical table—even a table compiled by the Army Medical Department. It is quite likely that several of Short's inferences were wrong. But he *did* paint a vivid picture of the changing conditions of life as he saw it. 'Colourless' is about the last adjective I should apply to his book. Had it been studied with more attention, had he been a leading London physician instead of an obscure country practitioner, medical statistics in England would have progressed faster.

VII. SOME REPRESENTATIVE CONTINENTAL DEMOGRAPHERS OF THE EIGHTEENTH CENTURY

My object is to sketch the history of distinctively medical statistics in our own country; I have neither the knowledge nor, perhaps, the desire to cover a wider field; but it would be too insular entirely to neglect continental research contemporaneous with that described in the preceding section. I propose to discuss the work of some foreign writers which is relevant to that of the British authors mentioned in the preceding section. The most eminent contemporaries of Short were Deparcieux, Wargentin, Struyck, Kerseboom and Süssmilch, and of these Deparcieux, Struyck and Süssmilch are, I think, the most interesting, a Frenchman, a Hollander and a German. None of them was a physician. Deparcieux and Struyck were competent mathematicians. Struyck wrote on the general theory of probability, Süssmilch had no more mathematics than Graunt; but, of the three, Süssmilch is better known to posterity because he is frequently cited in books which circulate outside professional statistical circles. Deparcieux (1703–68) is the least voluminous and most attractive of the three. He published in 1746 a quarto of 132 pages (with tables) entitled *Essai sur les Probabilités de la Durée de la Vie humaine*, to which he added, 14 years later, a short appendix, and his book is a model of clear writing.

Deparcieux was fully alive to the dangers of basing a life table upon data of mortality alone, and was the first writer to construct what we should regard now (subject to a few reservations) as correct tables. Of course, like his contemporaries, he could not make bricks without straw, and no more than they could provide a general population life table. He had to use data which were not random samples of human experience and is careful to point this out. His new material was drawn from two sources, the data of tontines and the mortality experience of religious orders.

A tontine (the name is derived from that of the inventor Lorenzo Tonti, a Neapolitan banker) was a system of selling annuities on the following plan. The participants are formed into age classes, each entrant pays a capital sum and receives an annuity; as the annuitants die out the amount payable to the survivors is increased and the last survivor will enjoy an income equal to that distributed originally over all members of the age class. This was the general plan of a simple tontine (*The Wrong Box* will have made us familiar with a different application); there were various modifications, but in all an exact record of deaths at ages was essential.

Deparcieux used the data of tontines established in 1689 and 1696. He had to face many difficulties. In the first place, the tontines had a series of classes, one for those entrants under the age of 5, the next for lives from 5 to 10, and so on. What is the mean age of the members of each class? There would, as Deparcieux points out, be a bias in the first class (that of children under 5) in

favour of ages beyond the mean, because parents needed no statistics to convince them that the rate of mortality in the first and second years of life is higher than in the third and fourth or fifth. In the later classes, on the other hand, the bias would be in favour of entering at an age below the mean of the class limits. He makes a rather modest allowance for these factors by taking the age at entrance in the first class as 3 years, i.e. half a year more than the mean of the class limits, and in the next (and subsequent classes) as half a year less than the mean. The next difficulty is that his observations end in 1742, consequently rates of mortality at ages are derived from persons whose dates of birth are widely separated. Thus no members of the first class of the 1689 tontine can have been exposed to the risk of dying at ages beyond 57 (actually of 202 entrants, 105 were still living at the close of the observations). So a table obtained by welding these observations ignores any secular trend of mortality. It also ignores what, in modern assurance practice, is an important factor, viz. selection. A life aged n years is less likely to end within the year of entrance than a life of the same age entered 10 years earlier. In ordinary practice there are two reasons, self-selection and medical examination. In annuitant experience only the former is involved, but this is not the *less* important of the two.

In the discussion of this subject which will be found in Elderton and Oakley's *The Mortality of Annuitants* 1900–1920 (published on behalf of the Institute of Actuaries in 1924), the conclusion is reached that when *contemporaneous* lives are in question, this selection only operates seriously on the first year of annuitant life; for that year the rate of mortality is about 63 % of that suffered by annuitants of the same age who had purchased annuities 5 years earlier (what is called ultimate mortality). If then there were no secular improvement of mortality rates—as there has been over the last 60 years—and if there were no secular change in the social or economic class of annuitants, while we should expect a lighter mortality upon recent entrants, if, as in Deparcieux's data, we are only given survivors at quinquennial intervals, we should not expect large differences. Actually one can test a particular age group, viz. 45–50 on numerically extensive material. The 1689 tontine provides ten and the 1696 tontine nine groups of persons of this age the survivors of which 5 years later are recorded. It will be seen from Table 1 that the 634 'new' entrants in the 45–50 tontine class of 1689 suffered rather heavier mortality than the 118 survivors to that age from the youngest class. This, however, is merely picking out a single pair. The correct test is to treat the data together and inquire whether the hypothesis that the whole set of deaths and survivorships might have arisen by sampling a population for which the chance of living 5 years was simply the ratio of total survivors to total exposed, viz. $5009/5394 = 0{\cdot}9286$. Applying the appropriate test, viz. that known as the χ^2 test (with 18 degrees of freedom), one reaches $P = 0{\cdot}0346$. This is not a very improbable freak of chance. Compared with modern annuitants, these tontiniers of 200 years ago had a rate

of mortality some 40% greater than the annuitants of 1900–20 between the ages of 47 and 52.

Finally, one has the class of society from which annuitants are drawn. Deparcieux was of opinion that annuitants were mainly drawn (op. cit. p. 62) from the middle class of society 'ce sont les bons Bourgeois qui tiennent un honnête milieu entre toutes ces extremités, qui se font des Rentes viagères; et ce sont ceux-là qui deviennent ordinairement vieux'. Hence he judged that the rate of mortality suffered would be less than that of the general population.

Table 1. *Deparcieux's observations*

Tontine class	Exposed to risk at age 47	Survivors to age 52		Deaths	
		Observed	Expected	Observed	Expected
(1689 tontine)					
– 5	118	109	110	9	8
–10	181	173	168	8	13
–15	211	192	196	19	15
–20	216	196	201	20	15
–25	201	189	187	12	14
–30	263	249	244	14	19
–35	526	479	488	47	38
–40	472	440	438	32	34
–45	770	723	715	47	55
–50	634	575	589	59	45
(1696 tontine)					
5–10	134	130	124	4	10
–15	131	118	122	13	9
–20	108	103	100	5	8
–25	102	92	95	10	7
–30	147	135	137	12	10
–35	211	204	196	7	15
–40	220	200	204	20	16
–45	444	415	412	29	32
–50	305	287	283	18	22
	5394	5009		385	

The next part of his investigation related to the mortality experience of members of monastic orders. These he utilized with the same good sense and care.

In Table 2 are his l_x values, to which I have added those for English Life Table No. 9 Males (general mortality of 1930–2). The reason for putting l_{20} equal to 814 is simply that in his table for tontines where the starting point is age 3, his survivors to age 20 from an initial 1000 were 814.

The column headed Benedictines (*a*) is a methodologically correct table, viz. based on entrants followed until death, Benedictines (*b*) assumes a stationary

population and is not therefore so exact although it utilizes more data. Actually both tables give virtually the same results. It will be seen that to age 50 all the tables agree well; after age 50 the monks fare worse than the members of tontines and worse than the nuns. All have much worse mortality than the unselected general population of England and Wales 200 years later. Deparcieux attributes to selection the equality of tontine and monastic mortalities at younger ages and to the privations and austerities of the religious life a higher mortality at later ages.

In an investigation made by Dr S. Monckton Copeman and myself some years ago (*Report on Public Health and Medical Subjects*, no. 36, H.M.S.O. 1926) into the alleged low mortality from cancer of members of certain religious orders, we had occasion to study the general mortality experience. The result was that the mortality at ages over 25 of monks was rather more favourable than that of

Table 2. *Deparcieux's observations*

Age	Tontines	Survivors from age 20		Nuns	E.L. No. 9 Males
		Benedictines (*a*)	Benedictines (*b*)		
20	814	814	814	814	814
30	734	756	749	751	788
40	657	675	681	676	755
50	581	575	583	587	698
60	463	423	432	462	594
70	310	236	235	286	405
80	118	55	51	103	151

annuitants, that of nuns less favourable. The data were, however, scanty (monks 65 observed against 79·4 expected deaths; nuns 152 observed against 124·7 expected deaths).

Deparcieux has a few remarks on general medical-statistical questions (for instance, he urges strongly the importance of mothers nursing their infants), but nothing of much significance.

The statistical writings of Nicholas Struyck (1687–1769) are more voluminous than those of Deparcieux* and cover a wider field. Struyck was the son of an Amsterdam burgher and is said to have been in relatively easy circumstances. He enjoyed a considerable reputation as a writer on mathematical, statistical, geographical and astronomical subjects and was admitted a fellow of the Royal Society of London in 1749.

* They were collected and published in French translation at the instance of the Netherlands Assurance Society in 1912: *Les Œuvres de Nicolas Struyck, qui rapportent au calcul des chances*, etc., traduites du Hollandais par J. A. Vollgraff, Amsterdam, 1912, pp. 430.

Struyck was evidently a competent mathematician and also an industrious field worker who carried out or inspired in the Netherlands many town and village enumerations of population and vital statistical records. Like Deparcieux, he constructed life tables from annuitants' data and he certainly understood the correct arithmetical procedure. His data were, however, much fewer and he does not give sufficient details of his methods of interpolation and approximation to central ages for it to be possible to say precisely how he reached the life tables for males and females printed on p. 231 of his book. The original data were 794 males and 876 females (annuitants) observed for various periods and classified in quinquennial age groups. One has the impression that, although Struyck was a mathematician, he was not very sensitive to the dangers of basing conclusions upon small absolute numbers, and in his discussion of the vital statistics of London (op. cit. pp. 348–51) he has hardly given enough weight to the disturbing influence of migration and is perilously near the fallacy of a stationary population.

From the point of view of the medical statistician, Struyck is not a very suggestive writer. As demographer, we might rank him as technically superior to Short but medically less interesting. Like his contemporaries he can chase phantom hares in a thoroughly entertaining way. His finest example is in a section on multiple births. After a sober statistical inquiry he concludes that a case of quintuplets might reasonably be expected to occur sometimes in populations of the sizes of those of France and Germany—'it would be a very rare but not an incredible event'.

The case of the countess of Hennenberg, alleged to have brought to birth 364 or 365 infants simultaneously, does, however, strike him as 'absolutely fantastic and contrary to nature', and he carefully examines the legend. The statement was that the prolific mother produced as many children as the days of the year, and that the boys were named John and the girls Elizabeth. As Struyck justly observes it would be silly to have 182 Johns and 182 Elizabeths, and by careful research he arrives at a simple rational solution. The lady performed her feat on 26 March 1266; at that period the year began with the Feast of the Annunciation which was 25 March. So the birthday was the *second* day of the year and probably the mother had twins, one christened John the other Elizabeth. *Simplex munditiis!*

The name of Johann Peter Süssmilch (1707–67) is far better known than those of Deparcieux and Struyck although it is doubtful whether his *book* is often read. The perusal of 1201 pages of text and 207 of tables (the contents of the third edition of Süssmilch's book, published in 1765) requires a powerful appetite. If Süssmilch's literary style has less complexity than that of successors who wrote after German had become a 'literary' language, it has not much charm and few of us love propaganda. Süssmilch is a pure propagandist; the title of his book is: 'Die göttliche Ordnung in den Veränderungen des

menschlichen Geschlechts, aus der Geburt, dem Tode und der Fortpflanzung desselben *erwiesen*' (italics mine), von Johann Peter Süssmilch. He sets out to reveal the divine machinery for fulfilling the command: 'Be fruitful, and multiply, and replenish the earth, and subdue it.' The reason why his book has more interest for a statistician than, say, Warburton's *Divine Legation*, is that Süssmilch conceived the notion that vital statistics might be pressed into the service of orthodox Lutheran theology, and the diligence with which he pursued his arithmetical investigations gave his book importance. It was indeed the quarry from which Malthus obtained material when the interest aroused by the first edition of his famous *Essay* led him to expand what had been not much more than a Shavian paradox into a serious treatise.

As a demographer and statistician, Süssmilch was technically inferior to either Deparcieux or Struyck and, of course, far below Halley. He had none of Graunt's originality and made no methodological advance. But he was very industrious. He assembled not only a large collection of German data, similar to but wider than those of Short, but collected foreign material—including that of Graunt, King and Short—and his tables are of real value.

The general conclusions he reached—constancy of the sex ratio, greater mortality of towns, etc.—differ in no important respect from those of his predecessors or English contemporaries. His own life table (which gave an expectation of life at birth of 28·43 years) is constructed on the incorrect principles adopted by most of his contemporaries. He was, indeed, aware that to make a life table by summing the deaths at ages occurring in an increasing population was wrong and that the population he used was increasing, but he did not know how to do better—indeed, he had no material for doing better.

His contribution to purely medical statistics is small. He has a chapter on the statistics of causes of death and compares the distribution by causes in the London Bills 1728–57 with those for Berlin in the years 1745, 1750 and 1757. When allowance is made for differences of nomenclature and misprints, the proportional distributions by causes are not very different. He makes the sensible suggestion that if the Latin names of the diseases were given by the medical attendants in official returns international comparison would be facilitated. For the rest, his medical importance is slight. To criticize or make fun of his triumphant justification of the ways of God to man would be sorry trifling. Although a dull writer, he inspires a certain affection. He was a sincere, diligent man and in polemics more courteous than most. He may, perhaps, quite contrary to his intention, have had a rather depressing influence upon enthusiastic readers, in that he had no expectation of a great reduction of mortality rates and has often anticipated ideas which we usually attribute to Malthus. He perceived that at the current rate of growth the earth must eventually be overpopulated, but he argued that as the density of population increased the age of marriage would rise and consequently the fertility rate would decline. 'If,

however, fertility remained the same, it would only be necessary for the rate of mortality to increase a little, so that, as in large towns, one in 25 died' (op. cit. **1**, 267).

He devotes a whole chapter to what Malthus would call positive checks upon population and clearly does not expect these to be eliminated although, for the reason just quoted, he does not think plagues and wars essential conditions. One, perhaps only one, item of the vital statistical system gave the good man some qualms. His arithmetic leads him to conclude that in cities half those born are dead by the 20th year of life, and even in the virtuous country districts half are dead before the age of 25. 'What is the reason that God permits half to die before they can be of service to God and the world? All the labour and effort of birth and rearing seem to have been in vain' (op. cit. **2**, 312). In a worldly sense there is, he confesses, no explanation. One must think of earthly life as but a preparation for the hereafter.

The trend of this reasoning is not encouraging to the social or hygienic reformer. Perhaps Süssmilch did contribute a little to the view that not much could be done to reduce the general death-rate, that, at the best, town death-rates might be slightly improved. But I doubt whether he had much influence upon medical opinion in England. Statistics are not even now a favourite study of the medical profession; 200 years ago a voluminous German writer on vital statistics would have found very few readers in the College of Physicians.

VIII. METHODOLOGICAL ADVANCES

The writers who were the subject of the last sections all flourished in the first half of the eighteenth century and all have a claim to be reckoned as pioneers. Deparcieux and Struyck made definite contributions to the mathematical or arithmetical technique of life-table construction; Süssmilch and Short followed the path blazed by Graunt, but they explored a good deal of country, and Short, at least, had novel ideas as to the utilization of local records.

In the later years of the century various medical writers, for instance, Heysham, Haygarth and Percival, made effective use of local enumerations of population in their efforts to secure sanitary improvements. The public has no passion for statistics, still a death-rate *is* more telling than a mere enumeration of deaths. But none of these writers contributed anything new to statistical methodology, and simple arithmetic, not to speak of the labour of making unofficial counts of population, is not every man's hobby. Had the proposal for making an official census in the middle of the century been accepted, no doubt interest in political or medical arithmetic would have revived, but it did not pass the House of Lords. The only official data of large dimensions were still the London Bills. These were sometimes the subject of medical statistical comment. In 1800, the younger Heberden wrote a monograph the title of which suggested competition with Short or even Graunt. But it was not a successful venture and

is only remembered now (if at all) because of a statistical 'howler' which the iconoclastic Charles Creighton exposed with a satisfaction not melancholy.*

A typical example of the attitude of the better class of physicians towards statistics at the end of the eighteenth century will be found in *Observations Medical and Political on the Small-Pox...and on the Mortality of Mankind at every Age in City and Country...*, by W. Black, M.D., the second edition of which appeared in 1781. Dr Black, a medical graduate of Leyden and a licentiate of the College, who survived to 1829, reprinted the life tables of his predecessors. He was alive to the importance of the statistical method and its neglect ('In the course of many years' attendance upon medical lectures, in different universities, I never once heard the bills of mortality mentioned', op. cit. p. 119) and held that 'the detached observations of physicians or other literary individuals, confined perhaps to a small town or parish: a meagre detail of village remarks (*sic*), afford in many instances a foundation too slight to erect upon them any general or permanent conclusions' (op. cit. p. 119). He accordingly devoted most of his attention to the London Bills, which he subjected to a severe but cogent criticism, and set out in detail a sensible plan for the compilation of data in London by salaried officials with medical knowledge, which, had it been adopted, would have antedated the establishment of effective registration in London by more than 50 years.

One might explain the stagnation of medical statistical research by saying that there was not enough straw for ordinary brick makers to be employed, and no medical man of sufficient ingenuity (or temerity) to find a substitute for straw emerged. If that eminent fellow and, for a very short space of time, president of the College James Jurin had lived in the second instead of the first half of the eighteenth century, it is possible that the history of medical statistics would have been different, because, some years after his death, two famous mathematicians tackled a problem in which Jurin had taken keen interest and, as he himself was an accomplished mathematician, their method would have given him intellectual pleasure.

Jurin was an enthusiastic supporter of the practice of smallpox inoculation and wished to provide an adequate statistical proof of its value. Monk provides an eulogistic, Creighton a depreciatory account of what Jurin did. A fuller account is given by Miss Karn (M. N. Karn, *Ann. Eugen.* **4** (1931), 279 et seq.).

That Jurin proved the fatality of inoculated smallpox to be very much less than that of the natural smallpox, even Creighton admitted. But that he did much more can hardly be claimed. Jurin virtually assumed that inoculated smallpox did confer an immunity, on the basis of others' testimony and the famous experiment on six criminals, or rather on the one criminal who after

* Creighton, *History of Epidemics in Britain*, **2**, 747. Heberden made two mistakes: (1) He did not recognize that 'Griping in the Guts' of the Bills of Mortality was mainly the Diarrhœa of young children. (2) That a gradual transfer from this heading to that of 'Convulsions' had been going on.

inoculation was deliberately exposed to natural infection (see Creighton, op. cit. p. 480). Whether Jurin deserves to be sneered at because he did not do what was impossible, or whether the assumptions he made were unreasonable, are questions I shall not discuss. The mathematicians added nothing to the biological discussion, the interest of their work is purely intellectual, viz. by showing how to make the most of imperfect material. The problem proposed by Daniel Bernoulli was this.

Let us assume that inoculation completely protects against dying from smallpox and that those who are thus saved from the smallpox are neither more nor less likely to die of other causes than persons who never take smallpox, then what would be the effect on general mortality of the total eradication of smallpox? Put more picturesquely, how many years would be added to the average span of human life if smallpox were extinct?

In modern times, questions like this have often been put and answered, because we know with fair accuracy the numbers living by sex and age and the numbers dying from different causes also by sex and age. In the famous Supplement to the *35th Annual Report of the Registrar-General*, Farr dealt with several causes. His method was simple. He subtracted from the central death-rate at any age due to all causes of death the central death-rate due to the special cause, and deduced from the resultant series of modified death-rates the appropriate life table constants. These he compared with those of the general life table. He found in this way that if phthisis were eliminated the expectation of life at birth (males) would be increased from 39·7 to 43·96 years. The elimination of the zymotic diseases would increase the mean lifetime to 46·77 years.

Farr was, of course, aware that the assumption, viz. if a particular cause of mortality was eliminated the death-rates from other diseases would not be affected, might not be justified—indeed, he had written with respect of Watt's lugubrious substitution theory, in accordance with which we gain little by eliminating one disease, its killing power will be taken by another. Farr's method is quite satisfactory as an arithmetical method but requires data not available in the eighteenth century. Bernoulli made two assumptions. The first that mortality rates from all causes were known (for his arithmetical calculations he used Halley's table although he did not quite correctly appreciate the meaning of Halley's phrase 'age current'), the second that the attack and fatality rates of smallpox were independent of age. He then reasoned thus:

Suppose there survive to age x by the life table P_x persons. Of these s, say, have not had smallpox; if $1/n$th of those who have not had smallpox were attacked within a year and $1/m$th of these die of smallpox, what is the value of s in terms of P_x, m and n? If dx is an element of time, $s\,dx/n$ will be attacked and $s\,dx/(mn)$ will die of smallpox within the element of time dx, and so there die

from other diseases $-dP_x - s\,dx/(mn)$ because $-dP_x$ is the total mortality. But we are only interested in s, so the decrement through mortality $-dP_x - s\,dx/(mn)$ must be multiplied by s/P_x, and we reach the equation

$$-ds = \frac{s\,dx}{n} - \frac{s}{P_x}\left(dP_x + \frac{s\,dx}{mn}\right),$$

the solution of which is $$s = \frac{mP_x}{(m-1)\,e^{x/n}+1}.$$

So s is known. Now let z be the number who would have survived to age x had there been no smallpox. Reasoning as before

$$-dz = -\frac{z}{P_x}\left(dP_x + \frac{s\,dx}{mn}\right).$$

The integral of which is $$z = \frac{P_x m\, e^{x/n}}{(m-1)\,e^{x/n}+1}.$$

This is the solution. Bernoulli put $n = m = 8$ and concluded that the elimination of smallpox would, on these assumptions, add about 3 years to the mean life-time.

D'Alembert criticized Bernoulli's assumption that m and n were constant and replaced his equation by the formally simpler equation

$$dz = \frac{z}{P_x}\,dP_x + \frac{z}{P_x}\,du,$$

where du is the increment of mortality in time dx due to smallpox. The formal solution is

$$z = P_x \exp\left[\int_0^x \frac{du}{P_x}\right]$$

Isaac Todhunter commented sub-acidly on this: 'The result is not of practical use because the value of the integral is not known. D'Alembert gives several formulae which involve this or similar unfinished integrations' (*History of the Theory of Probability*, p. 268). Todhunter's comment is just so far as concerns the situation when Bernoulli and D'Alembert wrote. If, in addition to a table of general mortality, one has knowledge of the deaths at ages due to smallpox, then by means of the theorem known as the Euler-Maclaurin expansion, it *is* possible to evaluate the integral and reach a solution on D'Alembert's lines as Miss Karn (op. cit. pp. 303 et seq.) has shown. But if we do have this information, the much less laborious method of Farr is adequate.

But that does not mean that the attempt of Bernoulli and D'Alembert was futile, a mere display of mathematical fireworks. The situation in which they found themselves recurs time and again in the history of statistics, indeed of all branches of science. Often a practical man objects that a mathematician will write down equations in general terms which cannot be solved and are therefore, as the practical man urges, of no use to him. Sometimes the practical man is

right, but not always; not even usually. Even when the equations cannot be solved, in the sense that certain 'constants' cannot be determined or certain integrals evaluated, methods of approximation, even inspired guesses, may lead to truth. Fifty years after Bernoulli and D'Alembert, E. E. Duvillard* published a monograph which, although seldom read, for it is scarce and 'practically' obsolete, has been rightly described by Farr as a classic of vital statistics. Duvillard set himself the same problem with the difference that vaccination was substituted for inoculation as the prophylactic, and this book, of nearly 200 quarto pages, may still be read with profit.

Duvillard lived before the days of Cauchy and mathematical rigour; no doubt much of his work would hardly satisfy the standard of a modern pure mathematician. Perhaps on that account it can be read by the amateur with comparative ease, and one may take hints of how to tackle problems for the solution of which complete statistical data are still to seek. There is no proverb the vital or medical statistician should more often repeat than the saying that the best is often the enemy of the good. It is no doubt foolish to suppose, as, according to Isaac Todhunter, Condorcet did suppose, that truth could be extracted from any data, however imperfect, provided one used formulae garnished with a sufficient number of signs of integration. It is more foolish to neglect even rough approximations to unattainable solutions. But, so far as concerns our predecessors in the College, indeed in the medical profession as a whole, the seed scattered by the foreign mathematicians fell upon stony ground. Between Short and Farr, no British physician made a contribution to statistical knowledge of much importance. I have spoken of the younger Heberden's brochure. William Woolcombe of Plymouth, in a tract on the alleged increase of tuberculosis, published in 1808, showed a better grasp of statistical method than the more famous physician.

The question Woolcombe examined was whether mortality from tuberculosis of the lungs were increasing. The statistical fact was that in well-kept registers he had examined the proportion of deaths assigned to consumption had certainly increased towards the end of the eighteenth century. Woolcombe was alive to the fact (often ignored by medical writers after his time) that the proportional mortality of a disease might increase although its absolute rate of mortality was stationary or even diminishing, and he tested his conclusions by a quite logical *ex absurdo* argument. Taking the assumption that at the beginning of the eighteenth century mortality was 1 in 36 and that the proportional mortality from phthisis was a third less than in 1801, he concluded that the general rate of mortality at the beginning of the nineteenth century must be as low as 1 in 54, unless the rate of mortality from phthisis had increased. But it was certain that in 1800 the general rate of mortality was higher than 1 in 54, at least 1 in 47. Reversing the process, viz. assuming the rate in 1801 to be known, the con-

* *Analyse et tableaux de l'influence de la petite vérole sur la mortalité à chaque âge*, Paris, 1806.

clusion was reached that the rate of mortality at the beginning of the eighteenth century must have been 1 in 27 unless the rate of mortality from phthisis had increased. This Woolcombe thought improbably high. He may have been wrong, but his method was rational. That was the best piece of medical statistical reasoning I have found in English medical literature between Short and Farr.

In 1800 the taking of a census was authorized by the legislature and not a government department, but the Speaker of the House of Commons was charged with the responsibility. Naturally, Mr Speaker passed over the actual work to one of his subordinates, and fortunately that subordinate, John Rickman, whose name is immortalized by the fact that he was a friend and correspondent of Charles Lamb, was really interested in statistics. In the report on the enumeration of 1801 comments are scanty, but they increased in subsequent volumes. Rickman was wholly responsible for the work down to the report of 1831 and, although he made no advance in statistical method, he did valuable work, particularly in calling attention to the high rate of mortality in the industrial north-west and in estimating past populations of the country. But Rickman was not professionally interested in medical questions, and before Farr no medical man utilized the new material effectively. As will be seen in the next section, the first English writer to publish a work under the title Medical Statistics was rather old fashioned in his treatment of the subject.

IX. THE END OF AN EPOCH

Almost at the end of the period I have chosen was published the first English book specifically devoted to Medical Statistics, *Elements of Medical Statistics*, by F. Bisset Hawkins, printed in 1829. It is a slender volume of 233 pages similar in format and size to the *Principles of Medical Statistics* published a little more than a century later, in 1937, by my friend and colleague Dr A. Bradford Hill.

Hawkins's book was an expansion of the Gulstonian Lectures of 1828; its author's long and useful life connects men still living with what seems a remote past. He was born in 1796, and there are still more than a dozen fellows of the College who may have sat in Comitia with him. He was admitted a fellow on 22 December 1826 and died in 1894. The copy of his book which I have read was presented by him to the Statistical (now Royal Statistical) Society in 1834 and contains corrections in his hand. Hawkins defines the province of Medical Statistics to be 'the application of numbers to illustrate the natural history of man in health and disease'. In his numerical statements he uses three indices; the ordinary crude death-rate—always expressed as one death in such or such a number—the 'probable life', i.e. the age to which half these born attain; the 'mean life', i.e. the average age at death. He was certainly aware that the age and sex constitution of a group affects the death-rate. Thus (op. cit. p. 20) he writes: 'In discussing the mortality of manufacturing towns or districts, it is just to remark that the small proportion is not always *real*; because a constant

influx of *adults* is likely to render the number of deaths less considerable than that which could occur in a stationary population composed of all ages.' From the use of the term *stationary population* in this passage we may also, perhaps, infer that Hawkins knew the limitations of utility of such indices as mean age at death or *vie probable*, but I cannot fairly say that in making comparisons he calls attention to the dangers.

A modern treatise, such as that of Dr Hill, devotes a large space to methods of evaluating errors of sampling or, to speak loosely, the precautions to be taken when the observations are few in number and may not have been taken without bias. Some of the methods still employed had been invented by mathematicians before Hawkins's day, but he did not use them. On p. 32 we read: 'The annual mortality of Nice, though a small town, and enjoying a factitious reputation of salubrity, is 1 in 31; of Naples, is 1 in 28. Leghorn is more fortunate, and sinks to 1 in 35. We instance those places as being the frequent resort of invalids; but how astonishing is the superiority of England, when we compare with these even our great manufacturing towns, such as Manchester, 1 in 74; such as even Birmingham, 1 in 43; or even this overgrown metropolis, where the deaths are only 1 in 40.'

In the copy I have read, the sentence 'such as Manchester, 1 in 74' has been struck through, apparently by the author. But, even with this emendation, the comparison, to the glory of our country, is, well, tendentious.

Indeed, one must admit, however regretfully, that Hawkins's book is uncritical. He has been diligent and brought together numerical data from all parts of the world and was certainly one of the first physicians to advocate a serious study of hospital records, but one can hardly say that, as a statistician, he was better equipped or more efficient than Dr Short in 1750. But his modesty is disarming: 'I should be amply rewarded if the present humble essay should form a temporary repository of the most important of their labours; if it should become one of the early milestones on a road which is comparatively new, rugged as yet and uninviting to the distant traveller, but which gradually discloses the most interesting prospects, and will at length, if I do not deceive myself by premature anticipation, largely recompense the patient adventurer' (op. cit. p. vii).

According to Munk (*Roll*, 3, 304) Hawkins was instrumental in obtaining the insertion in the first Registration Act of a column containing the names of the diseases or causes by which death was occasioned. 'At first the insertion was voluntary; it has since been made compulsory; and has produced important additions to medical and statistical science through the indefatigable labours of Dr W. Farr.'

So the name of Francis Bisset Hawkins deserves a place in the roll of benefactors to medical statistical science.

Eight years after the publication of Hawkins's Gulstonians there appeared,

as Chapter IV of the fifth part of McCulloch's *Statistical Account of the British Empire* (**2**, 567–601, London, 1837), an article on 'Vital statistics; or the statistics of health, sickness, disease and death', the work of William Farr, then in his 30th year and still a general practitioner and free-lance medical journalist. It contains perhaps a quarter of the number of words in Hawkins's book and is not free from the quaint moralizing not always wholly relevant to the statistical theme which was characteristic of Farr, but it ranks not much below Graunt's 'Observations' as an original contribution to medical-statistical science.

Farr proposed to examine 'the mortality, the sickness, the endemics, the prevailing forms of disease, and the various ways in which, at all ages, its [The British Population's] successive generations perish'.

Slow as had been the progress of official statistics between 1662 and 1837, there had been progress. The four censuses of 1801–31 provided reasonably complete accounts of total populations. In 1821, information as to age was invited and eight-ninths of the population accepted the invitation. In 1831 the clergy were asked to return not merely totals of burials but burials classified by ages for the 18 years ending in 1830. These latter returns were incomplete, but it was possible for a lesser man than Farr to approximate to a statement of rates of mortality at ages at least for the period centring on 1821. To Farr's annoyance, the census takers of 1831 did not ask for the ages of the enumerated, contenting themselves with an enumeration of males under and over 20 years of age. The data for computing mortality rates were particularly defective for towns, but a few instances of quite good voluntary enumerations, e.g. for Carlisle and Glasgow, were available.

In handling national rates of mortality at ages, Farr's article does not display any conspicuous originality; he, quite properly, used the work of predecessors and he does not comment on the defects of the data. He does, however, call attention to particular rates of mortality, for instance, those of the troops, in an emphatic way. 'By the subjoined table of the mortality of the British army it will be seen that the soldier, in the prime of his physical powers, is rendered more liable to death every step he takes from his native climate, till at last the man of 28 years is subject, in the West Indies, to the same mortality as the man of 80 remaining in Britain.' According to his table, the average strength of British troops in Jamaica and Honduras between 1810 and 1828 was 2528; in the year of least mortality the rate 47 per 1000, the average 113 and the maximum 472! In the United Kingdom the average rate was 15 per 1000.

The most original part of Farr's essay is his treatment of sickness. Here national statistics were not available; more than 70 years were to pass before any nation-wide data were collected, and the statistics of morbidity still lag behind those of mortality. All Farr had were some data of benefit societies and returns relating to workers in the Royal Dockyards and employees of the East India Company. He begins by stating that in manhood for every death

we may reckon two persons constantly sick. It is not quite clear how he reached this ratio, but probably from a comparison of the mortality rates for 1815–30 shown in a table on p. 568 of his article with some theoretical rates deduced by Edmonds for Friendly Societies (op. cit. p. 574). One has:

Age	Sickness rate per 1000	Mortality rate per 1000
20–30	17·2	10·1
30–40	23·0	11·4
40–50	31·0	14·9
50–60	45·1	23·4
60–70	93·6	45·3

Taking the general rate of mortality to be 21·3 per 1000 and the population of England and Wales to be 14,000,000, he concludes that 600,000 persons are constantly sick and that the productive power of the community is reduced by one-seventeenth part (he has made allowance for attendance on the sick). He works out from the limited data available the relation between sick-time and age and concludes that it increases in geometrical progression up to the age of 50. He asks how much sickness exists among the labourers of the country independently of those definitely incapacitated by disease. Data for the Royal Dockyards lead him to conclude that 2% are constantly kept at home by illness.

In the last section of his article, Farr considers particular diseases. An instance of his acumen is to be seen in his criticism of the view (held in 1837 as in 1937) that insanity was on the increase. He pertinently remarks that if the less barbaric treatment of lunatics diminished the mortality rate a higher proportion of enumerated lunatics would be perfectly consistent with a steady rate of morbidity.

His data for rates of mortality by causes were scanty. For London over a long period he had causes of death in age groups and, from an estimate of total mortality in age groups, could pass back to rates at ages by causes. Heysham's Carlisle data were medically and statistically more precise but limited to one not large town. The data of the Equitable Assurance Society were numerous but, as, of course, Farr knew and emphasized, related to a select class of the population.

Some of his general conclusions were as follows:

> It has been shown that external agents have as great an influence on the frequency of sickness as on its fatality; the obvious corollary is, that man has as much power to prevent as to cure disease. That prevention is better than cure, is a proverb; that it is as easy, the facts we had advanced establish. Yet medical men, the guardians of public health, never have their attention called to the prevention of sickness; it forms no part of their education. To promote health is apparently contrary to their interests: the public do not seek the

shield of medical art against disease, nor call the surgeon, till the arrows of death already rankle in the veins. This may be corrected by modifying the present system of medical education, and the manner of remunerating medical men.

Public health may be promoted by placing the medical institutions of the country on a liberal scientific basis; by medical societies co-operating to collect statistical observations; and by medical writers renouncing the notion that a science can be founded upon the limited experience of an individual. Practical medicine cannot be taught in books; the science of medicine cannot be acquired in the sick room. The healing art may likewise be promoted by encouraging post-mortem examinations of diseased parts; without which it is impossible to keep up in the body of the medical profession a clear knowledge of the internal change indicated by symptoms during life. The practitioner who never opens a dead body must commit innumerable, and sometimes fatal, errors (op. cit. p. 601).

Farr's article closes the epoch Graunt's book opened. The seventeenth-century pioneer did not live to see the ground he broke bear a crop. The high gods used Farr better; he lived to create the best official vital-statistics of the world. It is true that the lessons he taught were learned but slowly, either by the public or the profession. The *Annual Reports* of the Registrar-General will not be found among the frequently consulted volumes on the shelves of fellows of the College of Physicians. But something has been learned. The moral truism that human vanity is a deadly sin, now exemplified on a world-wide scale, is illustrated on the humbler scale of those topics which have been my life's work and the subject of these lectures. The distrust of 'mathematical' methods which is still general in our profession is not primarily due to the mere intellectual difficulty of learning 'mathematical' methods; much that all medical students must learn is at least as difficult.

The roots are deeper. They begin with the exaggerated claims of the iatro-mathematicians of the late seventeenth and early eighteenth centuries. The personal popularity of such men as Freind and Jurin did not conceal the fact that pathology and clinical medicine reduced to mechanical and quantitative theorems, and 'proofs' were of not much greater value in the treatment of sick men than skill in playing chess to the commander of an army. It is arguable that a talent for playing chess might, other things equal, be of advantage to a military strategist (Napoleon Bonaparte was very fond of chess and played so badly that it was difficult for his staff to avoid winning), but other things are not equal. In later times, when the intellectual prestige of mathematical science had grown enormously, it was observed that such an Admirable Crichton as our Thomas Young was inferior as a practical physician to many fellows of lesser fame. In our generation when the professional mathematicians who, 50 years ago, rather despised mere statistics, have increasingly devoted themselves to the improvement of the general theory, the complexity of statistical investigations has done little to attract the amateur, and intellectual modesty has not been the most conspicuous virtue of statistical authors. Perhaps, too, it is not easy for an experienced physician 'to renounce the notion that a science can be founded upon the limited experience of an individual'.

The moral I should draw from the history of medical statistics is that the intellectual courage of an amateur often succeeds where erudition fails. While even the purest of mathematicians would not claim that statistics is only a branch of mathematics, the hardiest contemner of algebra would admit that a training in mathematical method is an advantage to the practical statistician. The mathematician would surely agree that a knowledge of the material subjected to analysis was valuable, even if not so essential as a 'practical' man would claim.

Judged by contemporary intellectual standards, neither Graunt nor Farr was a mathematician; Graunt had no medical training, Farr's clinical experience was meagre. In respect neither of method nor subject-matter was either man an expert. But they both had intellectual curiosity and courage: one may say, if one pleases, the spurious courage of the man who is brave because he does not know what the dangers are. But, as Gilbert Chesterton once said, 'There is no real hope that has not once been a forlorn hope.' In graver matters than medical statistics and more than once in our national history salvation has been wrought by courageous amateurs who acted while professionals doubted.

Those who cannot disclaim a professional status in statistics, whether officials or professors, may learn a lesson from history. It is conveyed in the four words: *maxima debetur puero reverentia*, construing *puer* by amateur or beginner or enthusiast. It is weary work to read statistical 'proofs' of this or that aetiological theory of cancer, or proposals for this or that impossible statistical investigation. But it is treachery to science to rebuff any genuinely inquisitive person; the discovery of another Graunt in a shop or another Farr in the surgery of a general practitioner would repay the life-long boredom of all extant civil servants and professors of statistics.

A STUDY OF A SERIES OF HUMAN SKULLS FROM CASTLE HILL, SCARBOROUGH

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1. Introduction

Much of our knowledge of the early settlers of north-east Yorkshire, and more particularly of its medieval inhabitants, is derived from the archaeological material provided by Scarborough Castle Hill. Here, during the years 1921–5, a notable series of excavations, carried out under the supervision of Mr F. G. Simpson, M.A., Hon. F.S.A. (Scot.), revealed at three quite distinct levels an early Iron Age village, a Roman signal station, and three chapels, one earlier and two later than the Norman conquest.

The Iron Age remains, derived from a prehistoric occupation-layer associated with a series of forty-two or more rubbish pits and archaeological material, are doubtless a relic of those immigrants who arrived about the seventh century B.C. and sought a temporary dwelling-place on the headland between the landing-places of Scarborough, the present North and South Bays. Whether these settlers moved northwards along the coast towards the jet-producing region, or whether they were few in number and shortly vanished from the countryside can only be surmised. The fact that Castle Hill is almost the most northerly site at which evidences have been observed of active immigration from the Continent at this period suggests that their number could not have been large, although it is possible that a more numerous Iron Age population lived in the neighbouring Wolds.

The subsequent Roman occupation of Britain lasted until the early part of the fifth century A.D. when Roman troops were finally withdrawn from the country. The signal station which the excavations on Castle Hill revealed appears to have been built, like a number of similar coastal forts, at the instigation of the Roman general Theodosius, as a small garrisoned outpost against the Saxon raiders, some half-century before the final evacuation. Racially, therefore, the presence of this small body of men could have had little effect on the indigenous inhabitants of the district.

Thereafter, in the period preceding the arrival of the Norman conquerors, the surrounding district appears to have provided a battle-ground for Danes and foraying Norwegians. In 1066, what was probably still a small settlement at the site of present-day Scarborough was pillaged and its chapel and wooden houses were fired by the men of Tostig and Harald Hardraada. It is probable that this attack, and subsequent wasting of the north by William, practically extinguished the early township, and that it did not recover until the second quarter of the twelfth century, when William le Gros of Albemarle chose the site on the promontory to lay the foundations of his castle.

The rebuilding of the chapel took place after the foundation of the castle. It seems that its history after that period is shared largely with that of the castle, and we may assume

* The writer wishes to acknowledge the very substantial assistance given by Dr Morant in the composition of this paper.

that it was used by the population which gathered beneath the protection of the fortress. For a time the advowson of the church was the privilege of the owner of the castle, until it was given by Richard I to the abbot and convent of Cîteaux, in the diocese of Châlon-sur-Saône. The transfer of the beneficiary resulted in the establishment of a small community of foreign Cistercian monks, or 'custodes', which, however, probably never exceeded more than three in number. In subsequent years, owing to the outbreak of war with France, the connexion of this and similar alien priories with their parent houses was severed temporarily on more than one occasion. The association of the church with Cîteaux lasted, however, until 1400, when it was handed over to the prior and convent of Bridlington. The building retained its original use until 1538, but it was later adapted for domestic purposes (Rowntree, 1931).

This paper deals with the skulls which were recovered from the graveyards surrounding the post-Conquest chapels. Their medieval date may be regarded as established by a number of datable objects associated with the skeletons. These are a thirteenth-century pewter chalice, an engraved tomb cover, and a lead bullet found inside one of the skulls. There is evidence, also, of a post-mortem trepan on another of the skulls, which may be associated with the beginning of modern medical science in the mid-sixteenth century (see Tildesley, 1928). Except in one disputable area, the medieval graves lay about 2 ft. higher than the pre-Conquest remains, which were dated by a number of eighth-century jet crosses and the coin of Canute.

The skulls recovered from Castle Hill reposed, until recently, in the Museum of the Royal College of Surgeons, and they were catalogued by Miss M. L. Tildesley during the time she was Curator of Human Osteology there. Examination was made possible through the courtesy of Prof. A. J. E. Cave, the Assistant Conservator, to whom my sincere thanks are due for allowing me to study the collection. I am also glad to have this opportunity of acknowledging my indebtedness to Miss Tildesley for supplying various particulars regarding the skulls, and for her assistance in checking the dating of a number of them.

As the skeletons had previously been sexed by the late Dr Wingate Todd when he examined the long bones, it was decided to accept his estimates. The series comprises more than seventy skulls, but it was decided that measurements could only be used in the case of forty-three adult males, seventeen adult females and eight infants. Two specimens were rejected because they show evidence of posthumous deformity, and a further seven had reluctantly to be omitted owing to doubt regarding their medieval age. The majority of the crania were in excellent condition despite their long burial and removal from the ground; the somewhat battered state of the remainder made accurate measurement difficult in spite of skilful reconstruction.

The crania are mostly ovoid in the longer headed specimens, becoming ellipsoid in the broader headed examples. There is a well-marked 'roughness' in the nuchal area of some 30%, suggestive of strong neck muscles, and about as many specimens show fairly well-defined superciliary ridges. The zygomatic processes tend on the whole to be rather less noticeable. The mastoid processes and the occipital condyles are quite strongly developed, particularly so in a small number of cases, and the mandibles generally are strong and stoutly built. On the whole, therefore, the skulls denote a population possessing good muscular development. Morphologically, however, apart from the roughness of the nuchal area already noted, the most interesting feature (regarded in *norma lateralis*) is the quite definite occipital projection observed in many cases.

Mention has already been made of two cases of posthumously caused deformation, and in a further case the calvarial region had been so compressed as to render measurement impracticable. Few anomalous conditions were noted apart from ossicles in the region of the lambda.

2. THE TEETH OF THE SCARBOROUGH SKULLS

In view of the excellent state of preservation of the upper and lower jaws of the majority of the skulls it was possible to make a fairly detailed study of their dental and alveolar condition. Included in the fifty-six male and female jaws examined are most of the specimens omitted from the metrical study owing to doubt regarding their medieval authenticity. According to the archaeological evidence, however, they may be assumed no earlier in date than the eighth century A.D.

The analysis by age groups presented in Table 1 is based on estimates of age recorded by the late Dr Wingate Todd. As will be gathered from this tabulation, the average loss of teeth before death appears to have been very low, even for the older people. Assessment

Table 1. *Total numbers of teeth lost before death from groups of the adult Scarborough skulls (male and female)*

Approximate age groups (years)	No. of skulls	No. of teeth lost from		Average no. of teeth lost from both jaws
		Upper jaws	Lower jaws	
Under 30	8	—	—	—
30–40	11	2	6	0·7
40–50	16	28	12	2·5
50–60	18	86	49	7·5
Over 60	3	32	32	21·3
Totals	56	148	99	4·4

of teeth lost was based mainly on absorption of the alveolus, and it is probable that the average loss of 4·4 teeth per skull is a slight underestimate, since some teeth may have been lost shortly before death. In any case, the calculation must necessarily be crude, owing to the fact that some cases of third molars counted as lost may have been examples of a failure of the tooth to form. It was found that the first and second molars suffered most severely, and more particularly those in the upper palate. The loss of second premolars was about equal to that of third molars, while the anterior teeth were retained longer on the average.

Considered as a whole, the teeth were in fairly good condition, the most significant feature being marked attrition. As is usually observed, the wearing down of the crown is most marked in the case of the molars, the upper arcade being worn on the lingual side, and the lower on the labial, often to a remarkable angle. The teeth of the skulls in the younger groups were for the most part in very good condition, even up to the ages of 40–50, despite the fact that many of them were greatly eroded. Some half-dozen skulls carried large unerupted canines in both jaws, these being in one or two cases supernumerary to the full dental complement. In a fairly high proportion of cases there were signs of incipient

absorption of the alveolar ridge, indicative, presumably, of pyorrhoea. This feature did not appear to have affected the fixture of the relevant teeth to any particular degree. Protruding teeth were few, but in a small number of cases the incisors were prominently displayed owing to a slight degree of alveolar prognathism.

3. The variability of the Scarborough population

Standard deviations and coefficients of variation are given in Table 2 for the male series in the case of all characters for which thirty or more measurements are available. The numbers on which these constants are based (given in Table 3) range from thirty to forty-three. The female series is too short to give any estimate of variation worth considering. The male series may be supposed long enough to indicate any outstanding peculiarity of the population in the respect considered. It was only compared in detail with the Farringdon Street series of seventeenth-century London skulls (Hooke, 1926). This has often been used in estimating the relative variation of populations represented by cranial series, though it is rather less homogeneous than most of the others from British sites.

Table 2. *Standard deviations and coefficients of variation of the Scarborough male series (with probable errors)*

	S.D.	C. of V.
C*	94·3 ±7·2	6·23±0·48
L	8·05±0·59	4·34±0·32
B	5·50±0·41	3·77±0·28
B'	4·99±0·36	5·00±0·36
H'	5·22±0·38	3·92±0·29
S'_1	3·96±0·33	3·46±0·29
S'_2	6·87±0·57	6·03±0·50
S'_3	4·03±0·32	4·12±0·33
S_1	6·26±0·52	4·85±0·40
S_2	7·76±0·64	6·18±0·51
S_3	7·08±0·57	5·95±0·48
S	16·2 ±1·3	4·35±0·36

	S.D.	C. of V.
$\beta Q'$	11·8 ±0·87	3·73±0·27
U	16·7 ±1·2	3·17±0·24
fml	3·19±0·23	8·69±0·64
LB	4·21±0·33	4·18±0·33
GL	5·62±0·44	6·05±0·48
$G'H$	6·49±0·57	9·26±0·81
GB	4·71±0·40	5·03±0·42
NH, L	3·04±0·25	5·98±0·48
NB	1·66±0·13	6·67±0·51
O_1L	1·66±0·13	4·01±0·32
O_2L	2·92±0·24	8·46±0·69
G'_1	3·71±0·32	8·15±0·71

	S.D.
100 B/L	4·41±0·33
100 H'/L	4·13±0·30
100 B/H'	4·65±0·36
$Oc.I.$	4·06±0·33
100 NB/NH, L	3·26±0·27
100 O_2/O_1, L	7·10±0·57
$N\angle$	3°·56±0·31
$A\angle$	4°·70±0·41
$B\angle$	3°·67±0·32

* The constants are for reconstructed capacities.

Using coefficients of variation for absolute measurements and standard deviations for indices and angles, it is found that the Scarborough constant is the greater in the case of eighteen characters and the Farringdon Street is the greater in the case of the remaining fifteen. There are no markedly significant differences. The most significant are for C (Δ/P.E. $\Delta = 3{\cdot}9$), 100 NB/NH, L (3·2) and GB (3·0), for which the Farringdon Street constant is the greater, and $Oc.I.$ (3·9), $G'H$ (3·4), G'_1 (3·1) and 100 O_2/O_1, L (3·1), for which the Scarborough constant is the greater. Judging from all the characters, there appears to be no appreciable difference between the variabilities of the two populations. It may be noted that the standard deviation of the cephalic index for the Yorkshire series is unusually high, and it would have to be taken to indicate clear racial heterogeneity if found for a larger sample.

4. Comparisons between the Scarborough and other British series

Mean measurements of the Scarborough series are given in Table 3. The sex ratios of the absolute measurements are unexceptional, and in view of the small numbers of specimens the agreement between the corresponding male and female indices and angles is as close as

that expected for samples representing the same population. In such a case the mean female cephalic index is expected to be about one unit greater than the male index when the numbers are adequate, but no significance can be attached to the fact that a difference

Table 3. *Mean measurements of the Scarborough and two continental series of skulls**

	Scarborough		S.W. Norwegian‡	Belgian§
	Male	Female	Male	Male
C	1513·9 ± 10·2 (39)†	1365·0 (15)†	1467·5 (127)	—
L	185·5 ± 0·83 (43)	179·9 (16)	185·4 (144)	183·9 (133)
B	146·0 ± 0·59 (40)	139·9 (16)	143·6 (143)	145·3 (133)
B'	99·5 ± 0·51 (43)	97·4 (16)	97·9 (145)	98·2 (52)
H'	133·1 ± 0·54 (42)	126·7 (15)	129·3 (144)	131·1 (52)
S'_1	114·4 ± 0·46 (33)	113·1 (11)	112·5 (146)	—
S'_2	113·9 ± 0·81 (33)	112·4 (12)	112·0 (143)	—
S'_3	95·5 ± 0·46 (35)	96·9 (12)	96·6 (143)	—
S_1	129·0 ± 0·73 (33)	125·4 (11)	127·7 (146)	125·5 (52)
S_2	125·5 ± 0·91 (33)	122·7 (11)	124·8 (144)	127·3 (52)
S_3	118·9 ± 0·81 (35)	118·9 (12)	118·2 (143)	119·2 (52)
S	372·9 ± 1·9 (33)	366·2 (13)	370·2 (144)	372·1 (52)
$\beta Q'$	315·7 ± 1·2 (42)	302·5 (16)	308·6 (145)	—
U	525·9 ± 1·8 (41)	512·5 (16)	—	528·9 (52)
fml	36·7 ± 0·33 (43)	35·0 (16)	36·5 (139)	34·5 (33)
LB	100·6 ± 0·47 (36)	95·6 (14)	99·9 (142)	101·2 (33)
GL	92·8 ± 0·62 (37)	88·7 (14)	—	96·5 (33)
$G'H$	70·0 ± 0·80 (30)	63·4 (13)	72·2 (112)	—
GB	93·7 ± 0·56 (32)	87·0 (14)	94·7 (114)	—
J	133·2 (25)	122·4 (10)	134·0 (114)	132·0 (52)
NH, L	50·8 ± 0·35 (35)	47·8 (13)	52·5 (123)	—
NB	24·9 ± 0·18 (39)	23·4 (14)	24·6 (123)	23·5 (52)
O_1L	41·4 ± 0·19 (36)	41·0 (16)	41·3 (127)	—
O_2L	34·5 ± 0·33 (35)	35·7 (15)	34·5 (127)	33·6 (52)
G'_1	45·5 ± 0·46 (30)	43·7 (12)	46·0 (85)	—
G_2	40·6 (20)	35·6 (8)	41·3 (68)	—
$100\ B/L$	79·0 ± 0·47 (40)	77·6 (16)	77·6 (142)	79·0 (133)
$100\ H'/L$	71·8 ± 0·43 (42)	70·7 (15)	69·8 (141)	71·0 (52)
$100\ B/H'$	109·9 ± 0·50 (39)	110·3 (15)	{111·1 (143)}	{110·9 (52)}
$Oc.I.$	59·7 ± 0·46 (35)	58·5 (12)	{58·4 (143)}	—
$100\ G'H/GB$	74·7 (23)	74·8 (13)	{76·2 (112)}	—
$100\ NB/NH, L$	49·3 ± 0·39 (32)	49·4 (13)	47·0 (120)	—
$100\ O_2/O_1, L$	81·4 ± 0·81 (35)	87·8 (15)	82·2 (127)	—
$100\ G_2/G'_1$	91·1 (18)	81·6 (8)	90·6 (61)	—
$N\angle$	63°·5 ± 0·44 (30)	64°·4 (13)	—	—
$A\angle$	74°·7 ± 0·58 (30)	74°·1 (13)	—	—
$B\angle$	41°·8 ± 0·45 (30)	41°·4 (13)	—	—

* The mean indices in curled brackets were found from the means of the component lengths instead of from values for individual skulls.

† The capacities of the Scarborough skulls are estimates obtained by applying reconstruction formulae: see Appendix.

‡ Pooled means for the Bergen, Jaeren and Sogn series described by Schreiner (1939). Means for several other characters are provided in his monograph, and of these $fmb = 31·2$ (142), and $100\ fmb/fml = 85·8$ (139), are used in biometric practice in calculating coefficients of racial likeness.

§ Pooled means for series described by Heger & Dallemagne (1881). Other means are: Broca's $Q' = 311·2$ (52), $NH' = 50·1$ (52), $100\ NB/NH' = 47·1$ (52), $O'_1 = 39·0$ (52), $100\ O_2/O'_1 = 86·1$ (52), $fmb = 29·1$ (33), $100\ fmb/fml =$ {84·3 (33)}.

of the opposite sign is noted here. The mean female orbital index is normally appreciably greater than the male.

The female series must be considered too short to use for comparative purposes. In this

section comparisons by the method of the coefficient of racial likeness* are made between the male Scarborough and seven other British series, viz.:

(1) *Dunstable* (Dingwall & Young, 1933). The skeletons were found as secondary interments in a bell-barrow. The following remarks regarding their age are given in the report on the excavations (Dunning, Wheeler & Dingwall, 1931): 'The absolute dating of either of the two main series of burials (those in trench-graves and those buried superficially) is difficult. So close were they all to the surface of the mound that...it is impossible to affirm that any of the objects found in the same layer were, in the archaeological sense, associated with them....The general tenor of this evidence is that the surface of the mound was disturbed not later than the early Saxon period, and it is inferred that the burials (or the majority of them) are of the fifth or sixth century A.D.' Judging from cranial characters, however, the Dunstable is clearly distinguished from all Anglo-Saxon series.

(2) *Spitalfields* (Morant & Hoadley, 1931). This long series of skulls was recovered when excavations were carried out on the site of Spitalfields Market, London. Their age could not be ascertained as no datable artícles were found. The site formed part of a Roman cemetery, and it was almost certainly within the churchyard of St Mary Spittle (1197–1537). It was completely built over by 1688. Comparisons of the measurements show that the Spitalfields has its closest connexions with Pompeian and Etruscan series, while it is little further removed from the following.

(3) *Hythe* (Stoessiger & Morant, 1932). The skulls described form part of the large collection in the ambulatory of St Leonard's Church. The people probably died between A.D. 1100 and 1600. The series was found to have a closer connexion with the Spitalfields than with any other with which it could be compared.

(4) *English Bronze Age* (Morant, 1926). This series is believed to represent the Bronze Age invaders, the Neolithic element having been eliminated by a rough method. Revised means are given in *Biometrika*, **20** B (1928), 368–9.

(5) *Anglo-Saxon* (Morant, 1926). The skulls came from a number of cemeteries.

(6) *British Iron Age* (Morant, 1926). The pooled means represent the south of England better than the north, and England as a whole better than Scotland. The majority of the specimens are of Romano-British date. Revised means are given in *Biometrika*, **20** B (1928), 372–3.

(7) *Whitechapel* (Macdonell, 1904). The series came from a seventeenth-century London plague pit. A few additional measurements of the skulls are given in *Biometrika*, **18** (1926), 28–9.

There are data for other British series, but these seven were selected for comparison with the Scarborough because a preliminary comparison of the mean measurements showed that it diverges clearly from the prevailing type. The Moorfields and Farringdon Street, London, and the Glasgow Scottish series bear a close resemblance to the Whitechapel, and there is good reason to believe that these four represent the racial population which

* With the usual notation, the form of the crude coefficient is

$$\frac{1}{M}\Sigma\left(\frac{n_s n_{s'}}{n_s+n_{s'}}\times\frac{(m_s-m_{s'})^2}{\sigma_s^2}\right)-1\pm 0{\cdot}67449\sqrt{\frac{2}{M}}=\frac{1}{M}\Sigma(\alpha)-1\pm 0{\cdot}67449\sqrt{\frac{2}{M}}.$$

If $\overline{n}_s$ is the mean number of skulls available for the characters used in the case of the first series, and $\overline{n}_{s'}$ is the same for the second series, then the 'reduced' coefficient is defined to be

$$50\times\frac{\overline{n}_s+\overline{n}_{s'}}{\overline{n}_s\overline{n}_{s'}}\left\{\frac{1}{M}\Sigma(\alpha)-1\right\}\pm 0{\cdot}67449\sqrt{\frac{2}{M}}\times 50\times\frac{\overline{n}_s+\overline{n}_{s'}}{\overline{n}_s\overline{n}_{s'}}.$$

has been spread fairly uniformly over England and the south of Scotland in modern times. The British Iron Age series denotes a population which had very similar characteristics, and the Anglo-Saxon, while standing distinctly apart, was not far removed. The mean male cephalic indices for these six samples have the very restricted range from 74·0 to 75·5. The British Neolithic value is still lower, being 71·7. The Scarborough mean index is 79·0, which is not far removed from the Dunstable (78·7), Spitalfields (79·4), Hythe (82·6), or English Bronze Age (81·3). Racial affinities should not be judged from a single character, but these comparisons are sufficient to show that the Scarborough population cannot have been closely related to that prevailing in England in any period, except possibly the Bronze Age, while it may have been closely connected with the aberrant communities found at Dunstable, London (Spitalfields) and Hythe.

Table 4. *Coefficients of racial likeness between the Scarborough and other male series**

	Scarborough (34·9) and		
	Crude C.R.L.	Reduced C.R.L.	α's > 12
Dunstable (40·2)	0·63 ± 0·19 (26)	1·68 ± 0·51	—
Spitalfields (167·4)	3·68 ± 0·18 (28)	6·37 ± 0·31	$L = 20·9$, $O_2L = 16·8$, 100 O_2/O_1, $L = 13·1$, $C = 12·1$
Hythe (102·1)	5·18 ± 0·18 (28)	9·96 ± 0·35	$L = 42·6$, 100 $H'/L = 37·7$, 100 $B/L = 31·6$
English Bronze Age (31·1)	3·65 ± 0·19 (25)	10·98 ± 0·57	$O_1L = 31·1$, 100 O_2/O_1, $L = 14·3$, $B = 12·1$
Anglo-Saxon (35·9)	4·32 ± 0·18 (27)	12·26 ± 0·51	100 $B/L = 34·6$, 100 $B/H' = 21·4$, $B = 15·3$, $L = 15·3$
British Iron Age (55·7)	5·12 ± 0·21 (21)	12·40 ± 0·51	100 $B/L = 25·9$, $L = 21·6$, $B = 17·5$, 100 $B/H' = 12·1$
Whitechapel (93·3)	7·61 ± 0·19 (25)	14·75 ± 0·37	100 $B/L = 56·3$, $O_1L = 30·2$, $B = 24·9$, $L = 15·1$
Parisians: l'Ouest (76·7)	1·01 ± 0·24 (16)	2·00 ± 0·48	—
Belgians (66·6)	1·45 ± 0·25 (14)	2·96 ± 0·51	—
Etruscans (79·3)	2·22 ± 0·19 (24)	4·64 ± 0·40	$B' = 15·8$
S.W. Norwegians (126·0)	2·77 ± 0·19 (25)	5·06 ± 0·35	$H' = 18·3$, 100 $H'/L = 12·3$
Finns (120·1)	3·18 ± 0·22 (18)	5·55 ± 0·38	$O_2L = 22·8$, 100 $H'/L = 14·3$, $L = 13·0$
Pompeians (87·1)	3·10 ± 0·21 (20)	5·99 ± 0·41	$L = 14·6$
Parisians: Cité (57·2)	2·94 ± 0·22 (18)	6·49 ± 0·48	$fml = 12·4$

* The numbers in brackets following the designations of the series are the average numbers of skulls available for the characters used in computing the coefficients ($\bar{n}$'s). The numbers in brackets following the crude coefficients are the numbers of characters on which they are based. The Farringdon Street standard deviations were used in calculating the coefficients in this table.

Coefficients of racial likeness between the Scarborough and the seven other British series are given in the upper part of Table 4. The crude value with the Dunstable is only just significant (being 3·3 times its probable error), and the reduced value actually indicates closer resemblance than any found between pairs of three series of seventeenth-century London crania. No one of the characters used considered singly shows a difference which is clearly significant. The Scarborough series is seen to be much further removed from both the Spitalfields and Hythe, its divergences from them being appreciably greater than that between the two (reduced coefficient = 4·14). The English Bronze Age gives a rather higher reduced coefficient, and, as was anticipated, still more marked differences are found from the types of the Anglo-Saxon, British Iron Age and Whitechapel series. The distinctions in all these cases depend chiefly on the fact that a few of the characters compared show

markedly significant differences. The Scarborough type differs most clearly from the Spitalfields and Hythe in having a greater calvarial length (associated with marked differences in the height-length and cephalic indices in the latter case), from the English Bronze Age in having a smaller orbital breadth, and from the Anglo-Saxon, British Iron Age and Whitechapel in having a shorter calvarial length but greater breadth, so that its cephalic index is markedly higher than the values for them.

5. Comparisons between the Scarborough and non-British series

Measurements are available for numerous series of skulls representing populations of Western Europe in historical times. When a new series from the region is compared with these it is usually possible to find five or more close connexions, indicated by reduced coefficients of racial likeness less than 5·0. Only one value of this order could be found between the Scarborough and other British series, viz. that of 1·68 with the Dunstable. A rough comparison of the means suggested that it was only likely to give reduced coefficients less than 5·0 with seven European (and non-British) series. These are:

(8) *Parisians: l'Ouest* (Broca, 1873). *Le cimetière de l'Ouest* was opened in 1788 and closed in 1825. The means used are given in *Biometrika*, **23**, 232.

(9) *Belgians* (Heger & Dallemagne, 1881). Measurements are given of four series, viz. three of murderers whose skulls are preserved in the universities of Brussels, Ghent and Liege, respectively, and one of men who died at Brussels. The means of these short series are very similar and they were accordingly pooled, giving the values in Table 3 above. Standard deviations for the total 133 specimens are: $L = 5{\cdot}84 \pm 0{\cdot}24$, $B = 5{\cdot}37 \pm 0{\cdot}22$, $100\ B/L = 3{\cdot}74 \pm 0{\cdot}15$. These are all less than the corresponding constants for the Scarborough series (Table 2). In spite of this the pooling of all the Belgian specimens is not entirely satisfactory, and comparisons with the pooled means are only of provisional value.

(10) *Etruscans* (Schmidt, 1887). The means used are given in *Biometrika*, **20** B, 370.

(11) *South-west Norwegians* (Schreiner, 1939). The means given in Table 3 above were obtained by pooling measurements given for three series from Bergen, Jaeren and Sogn, respectively. These three are very similar in type, judging by the coefficients of racial likeness given by Schreiner, and they differ from the series he describes from south-east Norway in having higher mean cephalic indices and in other respects. The dates of the skulls range from the thirteenth to the nineteenth century.

(12) *Finns* (Morant & Hoadley, 1931, pp. 229, 232). Pooled means were obtained by taking the measurements of several short series given by K. Hällstén and other anthropologists. The majority of the specimens were assigned to the eighteenth and nineteenth centuries.

(13) *Pompeians* (Nicolucci, 1882; Schmidt, 1884). The pooled means used are given in *Biometrika*, **23**, 232.

(14) *Parisians: Cité.* The means used are given in *Biometrika*, **23**, 232. They were obtained from Broca's MS. catalogues. The series represents the population of Paris in the twelfth century.

Reduced coefficients of racial likeness between all pairs of the series 8, 10, 12, 13 and 14 listed above are given by Morant & Hoadley (1931, p. 234). They range from 2·44 to 6·91. Values between them and the Spitalfields series are given in the same table, and their range is from 3·54 to 8·57. The reduced coefficients between the same five series and the Hythe (Stoessiger & Morant, 1932, p. 198) range from 7·61 to 13·52.

Coefficients of racial likeness between the Scarborough and these seven non-British series are given in Table 4. The samples are all fairly adequate in size, but in every case a smaller or larger number of the characters used when possible in calculating the coefficient is not available for them. Hence the values given are only approximations to those which would be obtained for the complete set of characters, but close approximation is expected when the number is 20 or greater. All the reduced coefficients are seen to be appreciably lower than any found between the Scarborough and a series representing the prevailing population of England in any period. No resemblance with a non-British series is quite as close as that between the Scarborough and Dunstable, but the newly described population appears to have been more closely allied to several continental ones than to the other populations intrusive in Britain, viz. the Spitalfields and Hythe.

6. THE RACIAL RELATIONSHIPS OF THE SCARBOROUGH POPULATION

It is safe to infer from the archaeological evidence that the skulls described in this paper are those of people who died between the early twelfth and mid-sixteenth century. It is not known whether the interments were made at one particular time during this period, or whether they were dispersed over the whole of it. The cemetery was attached to a monastic house which was used up to the time of the Dissolution, and the records suggest that its alien inmates were always few in number. Men predominate but women and children were also buried in the site, and it is probable that the majority of these people were members of the laity of the town.

As far as can be judged from the short series, the men, women and children belonged to the same population. The nature of this group and its racial relationships have to be judged from the forty-three male skulls. This is a small sample for the purpose. Judging from constants for all the characters recorded, there is no appreciable difference between the variation exhibited by the Scarborough series and that of seventeenth-century Londoners who were buried in the graveyard at Farringdon Street. At the same time there is a clear suggestion of racial heterogeneity in the fact that the former has the high standard deviation of 4·41 for the cephalic index, which is known to be the character most likely to reveal exceptional mixture. The Scarborough series is certainly not ideal for comparative purposes, and certain conclusions of a broad nature are all that can be legitimately derived from it.

Comparisons of cranial characters considered singly, and of a number considered in conjunction by using the method of the coefficient of racial likeness, suggest the following conclusions. Considered as a whole and as an English series the Scarborough must be supposed aberrant. The majority of the people represented, if not all of them, cannot have belonged to any of the populations which prevailed in the country at different periods since Mesolithic times. It is possible that a few of them were members of one of those populations, but this cannot be demonstrated. Comparison with the other English series of skulls which represent alien communities shows that the Scarborough bears a close resemblance to the Dunstable (probably of fifth or sixth-century date), but it differs appreciably from both the Spitalfields and Hythe series.

Comparison with continental material shows that the Scarborough bears a close resemblance to several of the mesocephalic populations of Western Europe, a group which embraces French, Belgian and certain Italian series, and to which the inhabitants of south-west Norway and Finns can be assigned. The closest connexions—which are of the

same order as that between the Scarborough and Dunstable—are found with a modern Parisian and a modern Belgian series. Several of the characters used when possible in computing coefficients of racial likeness are, however, not available for these two. In view of this fact, and of the defects of the Scarborough series, it would be unwise to lay any stress on the conclusion that it bears a slightly closer resemblance to two particular foreign series than to others belonging to the same group.

A general conclusion which appears to be justified is that the majority of the individuals from Castle Hill described in this paper were descendants of an alien community. It is known from historical evidence that Scarborough was a base for Vikings, who probably occupied the site until the eleventh century. The only known settlement in England of Icelanders was also located there. Hence it is not unlikely that the alien community referred to was pre-Conquest and of Scandinavian origin. Alternatively, it might be suggested that the foreign element was introduced by immigrants from the Low Countries in medieval times. A third possibility is that the peculiarity of the Scarborough population was due to the survival in it of an element derived from the Bronze Age population of Britain, but the evidence does not favour this hypothesis.

The only head measurements of living Yorkshiremen available are those given by Beddoe & Rowe (1907) for a series of ninety subjects from the West Riding. The mean cephalic index, given by Buxton, Trevor & Blackwood (1939), is 78·7. The corresponding cranial index may be supposed to be 77, which is well below the Scarborough value of 79·0, though rather greater than the means for the seventeenth-century London series. It is possible that a population of the Scarborough type had an appreciable effect in determining the characteristics of modern Yorkshiremen.

APPENDIX

Table 5 of individual measurements of the Scarborough skulls

The measurements were taken in accordance with the customary biometric technique. Owing to an error in interpretation, the foraminal breadths (*fmb*) were inaccurate, and hence they and the indices (100 *fmb fml*) are not recorded. The letters denoting measurements are: C = capacity in cm.^{3} The values given were not found directly but from the reconstruction formulae involving L, B and H' given by Hooke (1926, p. 33). L = maximum glabella-occipital length (Martin 1). B = maximum breadth (M. 8). B' = minimum frontal breadth (M. 9). H' = basio-bregmatic height (M. 17). S_1' = chord nasion to bregma (M. 29). S_2' = chord bregma to lambda (M. 30). S_3' = chord lambda to opisthion (M. 31). S_1 = arc nasion to bregma (M. 26). S_2 = arc bregma to lambda (M. 27). S_3 = arc lambda to opisthion. S = arc nasion to opisthion (M. 25). $\beta Q'$ = transverse arc passing through bregma (M. 24). U = horizontal circumference through the ophryon and above the superciliary ridges (M. 23*a*). *fml* = basion to opisthion (M. 7). LB = basion to nasion (M. 5). GL = basion to alveolar point. $G'H$ = nasion to alveolar point (M. 48). GB = facial breadth between lowest points on zygomatic-maxillary sutures (M. 46). J = bizygomatic breadth (M. 45). NH, L = nasion to lowest point on margin of pyriform aperture on left side. NB = maximum breadth of pyriform aperture (M. 54). O_1L = maximum breadth of left orbit (M. 51).

$100\frac{NB}{NH}$
—
51·9
52·1?
—
—
53·5?
—
52·3?
51·0
—
51·5
48·1
56·7
51·2
56·2
49·7
45·8
48·9?
42·9
—
44·8
48·8
—
47·7
—
—
49·7
49·4
44·4
50·5
—
47·3?
45·1
45·1
50·5
46·5
47·3
48·4
52·2
51·6
49·8
—
46·8
47·5
51·8
47·3
52·8
—
44·1
—
50·6
—
49·8
44·4
—
54·1?
52·6
52·1
46·5
47·0

$100\frac{NB}{NH}$, L	$100\frac{O_2}{O_1}$, L	$100\frac{G_2}{G_1}$	$N\angle$	$A\angle$	$B\angle$	Remarks
—	—	—	69°	66°	45°	Sagittal suture obliterated. Most teeth lost before death
51·9	74·9	—	64°?	74°·5?	41°·5?	Sagittal and coronal sutures pearly obliterated
52·1?	88·8?	—	—	—	—	Sagittal suture obliterated
—	—	—	69°	62°	49°	Middle aged. Most teeth lost before death
—	—	—	—	—	—	Sagittal, coronal and lambdoid sutures obliterated
53·5?	80·3?	89·9?	—	—	—	Middle aged
—	—	—	57°?	83°?	40°?	Sagittal and lambdoid sutures partly obliterated
52·3?	80·3	105·7?	63°·5	81°	35°·5	Third molars erupting
51·0	73·1?	98·0?	—	—	—	Sagittal and lambdoid sutures obliterated
—	82·1	—	—	—	—	Young adult
51·5	77·8?	91·6	62°	76°	42°	Ageing
48·1	83·9	73·5	70°·5	69°·5	40°	Sagittal, coronal and lambdoid sutures obliterated
56·7	74·7	73·5	69°	79°	32°	Ageing. Edentulous
51·2	69·0	—	63°·5	73°·5	43°	Middle aged
56·2	82·7	—	62°·5	77°·5	40°	Sagittal suture obliterated
49·7	80·3	—	—	—	—	Ageing. Edentulous
45·8	81·4	88·1?	63°·5	74°·5	42°	Sagittal suture nearly obliterated. Lower canines misplaced
48·9?	82·6	93·5	61°·5	72°·5	46°·5	Young adult. Third molars erupting
42·9	93·7	—	61°	71°	48°	Middle aged
—	—	—	62°·5	78°·5	39°	Sagittal and lambdoid sutures obliterated
44·8	88·4?	—	61°·5	75°	43°·5	Middle aged
48·8	77·7	113·4?	63°·5	72°·5	44°	Sagittal, coronal and lambdoid sutures obliterated
—	—	—	—	—	—	Young adult
47·7	—	89·1	61°	82°·5	37°	Middle aged
—	74·0	—	—	—	—	Sagittal suture partly obliterated
—	87·2	93·5	—	—	—	Middle aged
49·7	—	94·1	59°·5	79°·5	41°	Young adult
49·4	98·5	93·8	64°·5	75°	40°·5	Sagittal suture closed
44·4	81·3	88·2	65°·5	75°·5	39°	Young adult
50·5	78·0	105·0	59°	74°·5	46°·5	Ageing
—	90·2?	—	—	—	—	Middle aged
47·3?	96·1	—	60°·5	72°·5	47°	Sagittal and lambdoid sutures closing
45·1	80·1	—	60°?	81°·5?	38°·5?	Ageing
45·1	80·8	—	60°	75°	45°	Sagittal suture nearly closed
50·5	78·3	90·3	68°	69°·5	42°	Middle aged
46·5	87·6	—	61°·5?	78°?	40°·5?	Ageing
47·3	83·6?	72·9?	66°·5	71°	42°·5	Sagittal suture partly closed
48·4	79·8?	—	—	—	—	Aged. Edentulous
52·2	82·8	—	—	—	—	—
51·6	84·8	85·7	70°·5	70°·5	39°	Ageing
49·8	76·1	—	65°·5	71°·5	43°	Middle aged
—	76·7	—	—	—	—	Ageing. Edentulous
46·8	61·7	—	60°	77°·5	42°·5	Middle aged
47·5	88·4	—	65°	71°	44°	Sagittal and lambdoid sutures nearly obliterated
51·8	80·7	73·2	68°·5	71°·5	40°	Middle aged
47·3	85·3	87·0	62°·5	70°·5	47°	—
52·8	81·1	110·6	65°·5	79°·5	35°	Young adult
—	—	—	—	—	—	Middle aged
44·1	90·0	79·6	68°?	69°?	43°?	Middle aged
—	—	—	—	—	—	Sagittal and lambdoid sutures closed
50·6	82·0?	—	67°·5	75°	37°·5	Ageing
—	94·8	—	—	—	—	Middle aged
49·8	97·5?	86·3?	65°	73°	42°	Young adult. Posthumously deformed
44·4	78·1	59·1	65°·5	73°	41°·5	Middle aged
—	84·1	—	—	—	—	Middle aged
54·1?	86·8	100·0	59°·5?	74°·5?	46°?	Young adult. Metopic
52·6	90·7	—	60°?	82°·5?	37°·5?	Middle aged
52·1	100·7	57·0	72°·5	68°·5	39°	—
46·5	87·5	—	61°?	78°?	41°?	Ageing
47·0	88·9	—	57°	77°·5	45°	Middle aged

Grave no.	Sex	C^*	L	B	B'	H'	S'_1	S'_2	S'_3	S_1	S_2	S_3	S
3	♂	1564	195·5	144·5?	97·5	132	—	123·2?	98·8?	—	135	134·5?	—
6	,,	1689	197	142·5	96·2	145	119·9?	124·9?	100·1?	138	141	120·5?	402?
7	,,	1435	187·5	136·5	101·4	132?	114·0?	119·4	92·0?	134?	125·5?	118?	378?
9	,,	1462	188	143	108·8	128·5	108·4	117·5?	101·8	122	119·5?	127?	369
10	,,	—	187	139·5	97·2?	—	—	—	—	—	—	—	—
12	,,	1572	180	150·5	101·6	138·5	—	—	96·7	—	—	116?	—
14	,,	1440	187·5	138	95·2	131	111·5?	113·6?	92·4	126?	125·5?	113·5?	365?
18	,,	1335	177·5	140·5	97·6	124·5	114·2	113·8	94·1	118·5	128	115	362
20	,,	1528	190·5	145·5?	99·5	131	114·0	115·5	95·2	134	128	118·5	381
21	,,	—	192·5?	—	94·8	138	—	—	94·1?	—	—	114·5?	—
23	,,	1554	182·5	152	106·8	133·5	110·5	112·5	94·7?	126·5	128?	114·5?	369
26	,,	—	184	—	96·1	125	—	—	—	—	—	—	354
27	,,	1457	172·5	146	93·3	136·5	110·2	104·2?	99·2	126	119?	111·5?	361
28	,,	1733	196·5	158	107·7	135	120·4	130·9	96·1	140·5	147	119	407
29	,,	1567	193	144·5	102·1	134	114·1	119·0	101·2	131	131	128·5	391
31	,,	1649	188·5	159·5	107·0	136	116·2	111·1	95·3	132·5	123	119·5	375
36	,,	1475	192·5	142	100·5	127·5	117·7?	110·0?	101·0?	133·5?	121·5?	129·5?	385
37	,,	1485	171·5	149·5	93·1	137	111·2	111·3	103·5	125	112·5	133	371
40	,,	1632	180·5	155	103·8	140	114·0	113·1?	97·3?	126·5	126?	117?	370?
43	,,	1577	183	147	104·4	140	115·1	—	—	130	—	—	381
45	,,	—	205	—	111·9	136·5	118·7	—	—	136·5?	—	—	408
48	,,	1517	189·5	148·5	102·7	128	116·9?	116·0	99·5?	135·5?	130·5?	113·5?	382
49	,,	1425	186·5	142	93·4	126·5	—	114·1	93·8	—	131·5	116	—
50	,,	1489	182·5	142·5	101·0	135·5	118·3	115·3	92·1?	127	126?	111·5?	365
51	,,	1533	198·5	143·5	95·2	128	—	115·9	100·2	—	126·5	124·5	—
59	,,	1346	189	136·5	95·9	121·5	—	—	—	—	—	—	—
60	,,	1580	177	154·5	101·4	138	114·0	111·1	91·8	129	123·5	107·5	362
61	,,	1408	175·5	140	90·3	134·5	119·0	109·8	90·6	124·5	123·5	106	354
63	,,	1441	174·5	149	100·9	130·5	118·2	95·8	99·0	124·5	106	119	350
68	,,	1523	184	142·5	99·0	138	117·2	104·3	102·1	136·5	126·5	120·5	383
69	,,	1632	197·5	155·5	104·2	127·5	—	117·2?	93·4?	—	127·5?	122·5?	—
70	,,	1296	172	140	97·0	124·5	105·4	—	—	115	—	—	353
71	,,	1448	179·5	143·5	97·8	132·5	113·1	111·6	96·4	128	124·5	109·5	362
72	,,	1523	181·5	151	102·8	132	108·2	101·1?	99·4?	123·5	110·5?	109·5?	343
73	,,	1559	190·5	144·5	98·2	135	121·9	110·6	104·0	135·5	118·5	125	379
75	,,	1404	182	142·5	103·0	127	114·6	116·3	106·2	127	117·5	130	375
77	,,	1583	194	146	100·2	133·5	116·3	120·0	100·8	136	132	122	390
78	,,	1466	184	141·5	95·2	133	112·2	—	—	129·5	—	—	381?
79	,,	1537	187	147	93·7	133	120·0	111·3?	97·5?	127	126?	116·5?	370
81	,,	1351	169·5	144·5	91·6	128·5	111·1	116·1	96·0	113	122·5	114	350?
123	,,	1533	180·5	148·5	92·6	136	115·9	111·9	100·0	134·5	123·5	120	378
131	,,	1602	185	148	98·2	140	—	119·2?	104·9?	—	133?	126·5?	—
132	,,	1689	185·5?	153?	106·1	143·5	114·0	—	—	131	—	—	—
1	♀	1250	171·5	134·5	92·6	124·5	107·8	110·6?	94·5?	121	120?	110?	351
5	,,	1357	183·5	138	99·3	125	111·4	118·9	90·3	128·5	124	113·5	366
8	,,	1459	188	142	112·8	129	112·4	114·9	102·0	130	126	128?	384
13	,,	1332	184·5	137	98·6	122·5	118·8?	115·6	—	127·5?	127·5	—	—
15	,,	1364	181·5	141·5	99·8	124	112·2	—	—	121·5	—	—	375
17	,,	1435	181	142·5	93·5	131	118·5	116·4	94·3	124	129·5	115	370
30	,,	—	182	142	101·0?	—	—	—	—	—	—	—	371?
32	,,	1502	190	142·5	92·1	131·5	113·2	113·1	97·2	131·5	122	119	373
33	,,	1248	175	136	86·1	120·5	—	113·0	100·1	—	—	124·5	—
34	,,	—	—	—	—	—	—	—	—	—	—	—	—
35	,,	1444	183	142·5	93·0	130·5	112·2	119·8	94·2	127	132·5	118	378
41	,,	1366	183	140·5	98·6	124	—	111·1	88·9	—	123	114·5	—
42	,,	1405	175·5?	145	103·7	129·5	—	—	102·9	—	—	126	362
52	,,	1328	174	138·5	94·2	128?	—	—	—	—	—	—	356
65	,,	1355	180·5	140	97·1	125	117·1	107·2	100·4	120	115·5	121·5	357
76	,,	1404	182	142·5	103·0	127	114·6	106·3	106·2	127	117·5	130	375
135	,,	1225	164	133	92·8	128·5	105·8	102·8	91·4	121·5	112·5	107	341

* Reconstructed capacities.

S'_1	S'_2	S'_3	S_1	S_2	S_3	S	$\beta Q'$	U	fml
—	123·2?	98·8?	—	135	134·5?	—	338	498?	36·9
119·9?	124·9?	100·1?	138	144	120·5?	402?	323	540	38·7
114·0?	119·4	92·0?	134?	125·5?	118?	378?	292	518	35·1
108·4	117·5?	101·8	122	119·5?	127?	369	311	531	38·8
—	—	—	—	—	—	—	301	500?	31·2
—	—	96·7	—	—	116?	—	322	526	34·4
111·5?	113·6?	92·4	126?	125·5?	113·5?	365?	310	522	35·6
114·2	113·8	94·1	118·5	128	115	362	304	506	35·2
114·0	115·5	95·2	134	128	118·5	381	—	—	39·2
—	—	94·1?	—	—	114·5?	—	316	—	34·8
110·5	112·5	94·7?	126·5	128?	114·5?	369	316	531	38·4
—	—	—	—	—	—	354	293	516	43·1
110·2	104·2?	99·2	126	119?	111·5?	361	319	499	37·8
120·4	130·9	96·1	140·5	147	119	407	337	559	39·1
114·1	119·0	101·2	131	131	128·5	391	315	536	38·7
116·2	111·1	95·3	132·5	123	119·5	375	342	548	35·0
117·7?	110·0?	101·0?	133·5?	121·5?	129·5?	385	309	540	25·0
111·2	111·3	103·5	125	112·5	133	371	315	505	31·7
114·0	113·1?	97·3?	126·5	126?	117?	370?	336	534	40·9?
115·1	—	—	130	—	—	381	324	530	36·2
118·7	—	—	136·5?	—	—	408	327	560?	33·1
116·9?	116·0	99·5?	135·5?	130·5?	113·5?	382	312	541	38·1
—	114·1	93·8	—	131·5	116	—	301	519	35·4
118·3	115·3	92·1?	127	126?	111·5?	365	302	521	38·9
—	115·9	100·2	—	126·5	124·5	—	306	534	38·0
—	—	—	—	—	—	—	298	525	36·0
114·0	111·1	91·8	129	123·5	107·5	362	332	532	41·8
119·0	109·8	90·6	124·5	123·5	106	354	307	501	35·3
118·2	95·8	99·0	124·5	106	119	350	311	511	41·1
117·2	104·3	102·1		126·5	120·5	383	319	529	38·9
—	117·2?	93·4?		127·5?	122·5?	—	319	557	37·9
105·4	—	—	112	—	—	353	298	499	33·1
113·1	111·6	96·4	128	124·5	109·5	362	317	515	36·0
108·2	101·1?	99·4?	123·5	110·5?	109·5?	343	315	531	38·1
121·9	110·6	104·0	135·5	118·5	125	379	316	534	39·1
114·6	116·3	106·2	127	117·5	130	375	312	529	34·1
116·3	120·0	100·8	136	132	122	390	317	538	37·1
112·2	—	—	129·5	—	—	381?	316	525	36·2
120·0	111·3?	97·5?	127	126?	116·5?	370	319	531	36·1
111·1	116·1	96·0	113	122·5	114	350?	315	495	35·2
115·9	111·9	100·0	134·5	123·5	120	378	325	527	34·7
—	119·2?	104·9?	—	133?	126·5?	—	323	525	40·9
114·0	—	—	131	—	—	—	328?	543	37·9
107·8	110·6?	94·5?	121	120?	110?	351	299	489	36·9
111·4	118·9	90·3	128·5	124	113·5	366	310	516	36·1
112·4	114·9	102·0	130	126	128?	384	309	529	35·1
118·8?	115·6	—	127·5?	127·5	—	—	298	522	37·1
112·2	—	—	121·5	—	—	375	301	513	31·9
118·5	116·4	94·3	124	129·5	115	370	302	514	36·4
—	—	—	—	—	—	371?	315?	518	36·1
113·2	113·1	97·2	131·5	122	119	373	304	531	37·8
—	113·0	100·1	—	—	124·5	—	286	495	35·9
—	—	—	—	—	—	—	—	—	—
112·2	119·8	94·2	127	132·5	118	378	306	518	32·0
—	111·1	88·9	—	123	114·5	—	300	521	33·6
—	—	102·9	—	—	126	362	312?	516	32·8
—	—	—	—	—	—	356	296?	498	36·3
117·1	107·2	100·4	120	115·5	121·5	357	295	518	37·9
114·6	106·3	106·2	127	117·5	130	375	312	529	34·1
105·8	102·8	91·4	121·5	112·5	107	341	295	473	31·6

Table 5. *Individual measurements of the Scarborough skulls*

βQ′	U	fml	LB	GL	G′H	GB	J	NH, L	NB	O_1L	O_2L	G'_1	G_2	$100\frac{B}{L}$	$100\frac{H'}{L}$	$100\frac{B}{H'}$	Oc.I.	100
338	498?	36·9	96·0	98·4	74·5	90·3?	—	—	26·2	—	—	—	—	73·9?	67·5	109·5?	53·9?	82·5
323	540	38·7	105·2	99·0?	73·8?	99·2	138·5	50·3	26·1	44·2	33·1	49·2	—	72·3	73·6	98·3	59·7?	74·4
292	518	35·1	94·1	—	—	85·4	122	48·2	25·1?	40·2	35·7?	45·3?	—	72·8	70·4?	103·4?	55·8?	—
311	531	38·8	102·9	107·9	87·8	99·3	—	55·4	—	—	—	52·8	—	76·1	68·3	111·3	57·2?	88·4
301	500?	31·2	—	—	—	99·2	137	—	24·6?	—	—	—	—	74·6	—	—	—	—
322	526	34·4	107·0	88·9?	—	—	—	48·8?	26·1?	41·6?	33·4?	41·6?	37·4	83·6	76·9	108·7	60·4?	—
310	522	35·6	108·5?	91·3?	71·8?	—	—	55·4?	—	—	—	—	—	73·8	69·9	105·3	58·2?	—
304	506	35·2	93·9	85·5	55·2	87·0	118·5?	41·9?	21·9?	39·7	31·9	36·7?	38·8	79·1	70·1	112·8	58·6	63·4
—	—	39·2	102·6	94·6	—	—	—	50·5	25·7	45·7	33·4?	45·0?	44·1?	76·3?	68·8	111·1?	57·3	—
316	—	34·8	—	81·5	—	—	125·0?	—	24·5?	40·7	33·4	—	36·2	—	71·7?	—	58·8?	—
316	531	38·4	102·2	93·1	71·3	92·5	140?	54·2	27·9	42·8	33·3?	47·2	43·2	83·3	73·1	113·9	59·4?	77·1
293	516	43·1	103·0	102·3	72·4	91·9	127·5	52·2	25·1	42·9	36·0	52·9	38·9	—	67·4	—	—	78·8
319	499	37·8	99·0	93·8	54·2	—	137·5	48·0	27·2	42·7	31·9	42·0?	—	84·6	79·1	106·9	69·1?	—
337	559	39·1	96·1	89·5	69·2	96·9	141	49·6	25·4	45·2	31·2	47·1	—	80·4	68·7	117·0	57·6	71·8
315	536	38·7	101·0	91·3	66·8	93·5	138·5	49·1	27·6	42·1	34·8	41·0	—	74·9	69·4	107·8	56·2	71·4
342	548	35·0	106·7	—	—	94·9	142	52·3	26·0	43·1	34·6	—	—	84·6	72·1	117·3	56·9	—
309	540	25·0	103·4	95·8	72·2	97·0	—	52·2	23·9	42·0	34·2	45·3?	39·9	73·8	66·2	111·4	55·8?	74·3
315	505	31·7	92·2	85·2	71·0	—	129	51·1	25·0?	40·9	33·8	44·9	42·0	87·2	79·9	109·1	55·8	—
336	534	40·9?	97·8	87·8	76·9	—	—	55·2	23·7	41·2	38·6	47·6	—	85·9	77·6	110·7	59·7?	—
324	530	36·2	103·3	92·8	66·7	88·4	—	49·1	—	—	—	—	—	80·3	76·5	105·0	—	75·5
327	560?	33·1	104·9	95·9	75·0	—	138	53·8	24·1	44·8	36·9?	46·0	—	—	66·6	—	—	—
312	541	38·1	99·3	93·2	73·6	102·4	141·5	52·0	25·4	46·2	35·9	43·9	49·8?	78·4	67·5	116·0	66·3?	71·7
301	519	35·4	—	—	—	—	—	—	—	—	—	—	—	76·1	67·8	112·3	57·7	—
302	521	38·9	102·8	91·8	63·2	—	—	46·1	22·0	—	—	43·9	39·1	78·1	74·2	105·2	59·2?	—
306	534	38·0	—	101·1	—	95·6	—	—	25·4	41·2	30·5	—	37·1	72·3	64·5	112·1	57·4	—
298	525	36·0	—	93·9	—	95·7	—	—	26·7	39·9	34·8	46·2	43·2	72·2	64·3	112·3	—	—
332	532	41·8	102·1	90·0	69·1	92·7?	—	52·9	26·3	41·2	—	45·4	42·7	87·3	78·0	111·9	62·6	74·5
307	501	35·3	102·9	96·2	70·2	101·1?	134	49·8	24·6	41·4	40·8	43·5	40·8	79·8	76·6	104·1	62·6	69·4
311	511	41·1	97·8	92·1	64·5	91·4	—	49·1	21·8	43·2	35·1	43·2	38·1	85·4	74·8	114·2	59·9	70·6
319	529	38·9	101·3	90·2	76·2	91·1	135	50·9	25·7	42·2	32·9	42·0	44·1	77·4	75·0	103·3	61·7	83·6
319	557	37·9	—	98·8	—	97·6?	143	—	23·2?	44·1	39·8?	48·3	—	78·7	64·5	122·0	54·9?	—
298	499	33·1	94·5	86·2	73·1	—	—	52·4	24·8?	43·1	41·4	—	41·1	81·4	72·4	112·4	—	—
317	515	36·0	103·2	90·5?	66·1?	87·1	132·5	50·8	22·9	40·8	32·7	42·9	—	79·9	73·8	108·3	67·0	75·9?
315	531	38·1	104·8	94·1	78·0	94·6	133	59·8	27·0	44·7	36·1	44·7	—	83·2	72·7	114·4	73·9?	82·5
316	534	39·1	99·8	99·2	70·9	100·0?	136	49·7	25·1	43·4	34·0	44·2	39·9	75·9	70·9	107·0	59·9	70·9
312	529	34·1	94·2	84·1?	63·3?	83·2	120·5	48·2	22·4	42·6	37·3	—	—	78·3	69·8	112·0	58·4	76·1
317	538	37·1	98·6	95·7	71·0	89·8	132·5	49·9	23·6	43·2	36·1?	50·9	37·1?	75·3	68·8	109·4	59·3	79·1
316	525	36·2?	91·6	—	—	89·6?	126	46·9	22·7	40·1	32·0?	—	—	76·9	72·3	106·4	—	—
319	531	36·1	108·7	—	—	95·2?	—	52·3	27·3	43·1	35·7	—	—	78·6	71·1	110·5	60·4?	—
315	495	35·2	96·6	96·9	65·5	91·6	126·5	47·9	24·7	40·8	34·6	43·4	37·2	85·3	75·8	112·5	61·1	71·5
325	527	34·7	99·1	94·8	66·1	97·8	136	50·0	24·9	42·2	32·1	—	—	82·3	75·3	109·2	60·0	67·6
323	525	40·9	—	83·9	—	93·4	—	—	23·8	44·6	34·2	44·8	—	80·0	75·7	105·7	59·6?	—
328?	543	37·9	103·4	91·3	72·5	90·9?	—	53·2	24·9	43·1	26·6	53·9	—	82·5?	77·3?	106·6?	—	79·9?
299	489	36·9	92·3	88·1	68·1	86·4	122	46·7	22·2	39·8	35·2	44·5	—	78·4	72·6	108·0	63·3?	76·8
310	516	36·1	97·0	94·9	67·2	84·8	126	46·7	24·2	41·9	33·8	45·6	33·4	75·2	68·1	110·4	56·6	79·2
309	529	35·1	94·3	88·7	74·3	92·8	122	56·6	26·8	44·2	37·7	44·6	38·8	75·5	68·6	110·1	56·8?	80·1
298	522	37·1	94·8	88·0	55·8	85·4	—	42·8	22·6	39·8	32·3	36·8	40·7	74·3	66·4	111·8	—	65·3
301	513	31·9	—	—	—	—	—	—	—	—	—	—	—	78·0	68·3	114·1	—	—
302	514	36·4	95·7	94·8?	70·2?	89·3	125·5	50·6	22·3	40·9	36·8	48·1	38·3	78·7	72·4	108·8	58·7	78·6?
315?	518	36·1	95·7	—	—	—	130·5	—	—	41·9?	—	—	—	78·0	—	—	—	—
304	531	37·8	103·8	98·4	65·2	89·2	—	49·2	24·9	41·2?	33·8?	46·5	—	75·0	69·2	108·4	58·4	73·1
286	495	35·9	—	79·0	—	84·1	119·5	—	21·9	38·7	36·7	—	—	77·7	68·9	112·9	57·3	—
—	—	—	95·3	90·8	66·8	85·7	—	48·4	24·1	40·2	39·2?	45·1?	38·9	—	—	—	—	77·9
306	518	32·0	94·9	89·7	66·5	89·1	—	49·1	21·8	42·1	32·9	46·2	27·3	77·9	71·3	109·2	56·9	74·6
390	521	33·6	—	—	—	—	—	—	—	42·1	35·4	—	—	76·8	69·7	113·3	55·2	—
312?	516	32·8	90·0	80·5?	60·8?	89·0	—	45·8?	24·8	40·3	35·0	39·9	39·9	82·6?	73·8?	112·0	58·4	68·3?
296?	498	36·3?	99·2?	85·9?	61·2	88·0?	118·5	47·7	25·1	40·7	36·9	41·1	—	79·6	73·6?	108·2?	—	69·5?
295	518	37·9	97·3	99·7	66·0	90·2	125·5	44·9	23·4	40·8	41·1	47·9	27·3	77·6	69·3	112·0	59·3	73·2
312	529	34·1	94·2	84·1?	63·3?	83·2	120·5	48·2	22·4	42·6	37·3	—	—	78·3	69·8	112·2	58·4	76·1?
295	473	31·6	93·4	80·2	64·2	81·2	113·5	44·9	21·1	35·1	31·2	38·4	—	81·1	78·3	103·5	62·6	79·1

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O_2L = maximum height of left orbit (M. 52). G'_1 = length of palate from orale to staphylion (M. 62). *Oc.I.* = occipital index which is

$$100\,\frac{S_3}{S'_3}\sqrt{\left(\frac{S_3}{24\,(S_3-S'_3)}\right)}.$$

Values were found with the aid of a table of the function given in *Biometrika*, **13**, 261. $N\angle$, $A\angle$ and $B\angle$ are the angles of the triangle of which the nasion, alveolar point and basion are the apices. They were found from the sides $G'H$, GL and LB with the aid of a trigonometer.

The mean cephalic index for seven juvenile skulls was found to be 80·7.

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A STUDY OF THE CHINESE HUMERUS

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1. *Introduction.* In general the published anthropometric data relating to the humerus are far less extensive than those for the cranium, and this is particularly so in the case of Chinese material. Several good series of Chinese crania have been described (see Woo & Morant, 1932), but there are few memoirs which deal with series of Chinese skeletons in any detail. The material treated in Black's study (1925) comprises two short series from late prehistoric sites and one representing the modern inhabitants of the north China plain: all three relate to the northern part of the country. I have been able to study two series of skeletons from central China. One of these is short and of medieval date. The other is modern, and it is long enough to serve most statistical purposes. The present paper provides a description of the humeri, and it is hoped that detailed studies of the other bones of the skeletons will be published later. The modern material was extensive enough to make possible a more thorough statistical examination of bilateral and sexual differences than any which appears to have been provided previously for the humerus. Comparisons with other racial series were restricted by the fact that the writer had limited access to the literature while working under difficult war conditions.

2. *The new material.* Two series of Chinese humeri preserved in the Museum of the Institute of History and Philology, Academia Sinica, are described in this paper. The writer is greatly indebted to the authorities of the Institute, and particularly to Professors Fu Ssûnien and Li Chi, for granting him permission to study these bones. The adult specimens only were examined and they were sexed anatomically, with the aid of the crania and other parts of the skeletons. Most of the humeri are well preserved, but one or both extremities are defective in a few cases. The series came from:

(i) Hsiao T'un, Anyang. Skeletal remains representing the Sui-T'ang dynasties (A.D. 581–899) were excavated from 1929 to 1932 by Dr Li Chi and other Fellows of the Archaeological Section of the Institute. The specimens came from several cemeteries, and an account of the excavations will be given in a report on the crania which the writer is preparing. There are eleven pairs of male and seven pairs of female humeri. Measurements of unpaired bones in the two series were taken but they are not used in this paper.

(ii) Hsiu Chiu Shan, south of Hsia Kuan, Nanking. The skeletons are modern and they were obtained by the writer in 1936 from unclaimed graves used by the poorer classes. It is known that some of the people came from the eastern part of China and some are from unknown localities. There are seventy pairs of male and forty-three pairs of female humeri.

3. *The measurements recorded.* Martin's technique (1928) was followed and a selection of the measurements which appear to be most useful for comparative purpose was made. They are:

(1) Maximum length from the highest point on the head to the lowest point on the trochlea, measured with the osteometric board (M. 1).

(2) Total length from the highest point on the head to the lowest point on the capitulum, taken with the axis of the shaft parallel to the side wall of the board (M. 2).

(3) Breadth of the proximal epiphysis from the most medial point on the articular surface of the head to the most lateral point on the greater tuberosity, taken on the board with the shaft of the bone parallel to the side wall (M. 3).

(4) Breadth of the distal epiphysis, i.e. the maximum breadth between the condyles taken on the board with the bone in the same position as for (3) (M. 4).

(5) Maximum diameter at the middle of the shaft (determined from the maximum length (1)) taken in any direction with small callipers (M. 5).

(6) Minimum diameter at the middle of the shaft taken without reference to the direction of the maximum diameter with small callipers (M. 6).

(7) Circumference at the middle of the shaft measured with a steel tape (M. 7*a*).

(8) Minimum circumference measured with a steel tape and usually found at the second third, distal to the deltoid eminence (M. 7).

(9) Maximum sagittal diameter of the head from the highest point on the margin of the articular surface of the head to the lowest point on the same margin, taken with small callipers in a plane parallel to the long axis of the bone (M. 10).

(10) Maximum transverse diameter of the head perpendicular to (9) (M. 9).

(11) Circumference of the head measured round the margin of the articular surface with a slip of paper (M. 8).

(12) Angle of torsion. With the bone in the standard horizontal position, this is the angle between lines representing the axes of the articular surfaces of the proximal and distal ends projected on to a

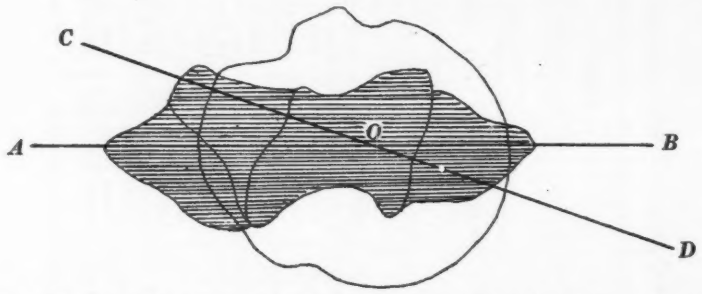


Fig. 1. Contours of the two ends of a humerus superposed, showing the angle of torsion ($< AOD$) between their axes. (*After Martin.*)

vertical plane. Fig. 1 illustrates the way in which it is measured. The line CD is the axis determined by the 'centre' of the head and the 'mid-point' of the greater tuberosity, so that it appears to divide the outline of the proximal extremity into two equivalent areas. The line AB is the axis of the distal extremity determined so that it appears to halve the surface of the trochlea and capitulum which is seen. The two lines intersect at O and the angle AOD is defined to be the angle of torsion (M. 18). Some authors use the supplement of this angle instead. In practice the bone is held vertical by a clamp, and two fine knitting needles are used to represent the axes. Their directions are marked on a sheet of paper with the aid of a parallelograph and the angle between them is measured with a protractor (M. 18).

Three indices derived from pairs of the direct measurements defined above were used, viz.:

(13) Cross-section index of the diaphysis = 100 × 6/5.

(14) Caliber index = 100 × minimum circumference (8)/maximum length (1).

(15) Head index = 100 × 10/9.

The measurements taken on the board and the circumferences were recorded to the nearest 0·5 mm., and the chords, found with small callipers provided with a vernier scale, were recorded to the nearest 0·1 mm. Constants derived from the measurements are given in Table 1.

4. *Bilateral comparisons.* For the eleven measurements of size the mean on the right side exceeds the corresponding value on the left in the case of both male and female series with only one exception, viz. the minimum diameter at the middle of the shaft for the female Hsiu Chiu Shan series. It had been found from material representing other races that on the average the right humerus is appreciably larger than the left in nearly all respects.

Table 1. *Constants (with probable errors) for two Chinese series of humeri: paired bones only*

Constant...	Means							
Sex	Male				Female			
Series	Hsiao T'un		Hsiu Chiu Shan		Hsiao T'un		Hsiu Chiu Shan	
Side	R.	L.	R.	L.	R.	L.	R.	L.
1. Maximum length	312·9 (7)	308·9 (7)	310·32 ± 1·25 (61)	306·72 ± 1·20 (61)	286·4 (5)	282·6 (5)	284·94 ± 1·81 (38)	281·71 ± 1·74 (38)
2. Total length	307·0 (7)	304·0 (7)	304·50 ± 1·20 (60)	301·88 ± 1·16 (60)	278·9 (5)	276·5 (5)	278·45 ± 1·76 (36)	276·16 ± 1·68 (36)
3. Br. prox. epiphysis	48·5 (6)	47·9 (6)	48·03 ± 0·19 (59)	47·81 ± 0·20 (59)	43·8 (5)	43·3 (5)	43·53 ± 0·32 (36)	43·18 ± 0·29 (36)
4. Br. dist. epiphysis	59·8 (6)	59·2 (6)	59·40 ± 0·23 (64)	58·79 ± 0·22 (64)	54·3 (7)	53·5 (7)	53·98 ± 0·44 (38)	53 24 ± 0·42 (38)
5. Max. diam. mid. shaft	23·4 (11)	22·6 (11)	22·37 ± 0·09 (61)	21·65 ± 0·09 (61)	20·0 (7)	19·0 (7)	19·97 ± 0·16 (38)	18·99 ± 0·15 (38)
6. Min. diam. mid. shaft	18·1 (11)	17·6 (11)	17·15 ± 0·11 (61)	17·01 ± 0·10 (61)	13·9 (7)	13·5 (7)	13·62 ± 0·12 (38)	13·65 ± 0·11 (38)
7. Circum. mid. shaft	68·5 (11)	66·9 (11)	67·81 ± 0·43 (61)	66·24 ± 0·45 (61)	58·5 (7)	56·9 (7)	57·41 ± 0·46 (38)	55·93 ± 0·46 (38)
8. Min. circum. shaft	64·8 (11)	63·1 (11)	63·98 ± 0·38 (70)	62·58 ± 0·35 (70)	56·5 (7)	55·0 (7)	56·04 ± 0·37 (42)	54·53 ± 0·38 (42)
9. Sag. diam. head	46·0 (7)	44·0 (7)	46·08 ± 0·18 (65)	44·09 ± 0·18 (65)	39·6 (6)	38·4 (6)	39·33 ± 0·21 (43)	38·21 ± 0·22 (43)
10. Trans. diam. head	43·8 (7)	42·8 (7)	43·64 ± 0·19 (64)	42·70 ± 0·17 (64)	36·5 (6)	36·0 (6)	36·31 ± 0·19 (43)	35·82 ± 0·19 (43)
11. Circum. head	138·2 (7)	136·5 (7)	137·56 ± 0·53 (65)	135·96 ± 0·53 (65)	123·0 (6)	122·1 (6)	121·69 ± 0·77 (43)	120·82 ± 0·75 (43)
12. Angle of torsion	149°·3 (6)	148°·6 (6)	149°·30 ± 0·73 (61)	148°·81 ± 0·70 (61)	146°·2 (7)	145°·6 (7)	146°·95 ± 0·84 (38)	146°·17 ± 0·71 (38)
13. Index of diaphysis	77·1 (11)	78·2 (11)	76·66 ± 0·29 (61)	78·54 ± 0·28 (61)	68·9 (7)	70·8 (7)	68·28 ± 0·25 (38)	71·62 ± 0·30 (38)
14. Caliber index	20·8 (7)	20·4 (7)	20·72 ± 0·15 (61)	20·34 ± 0·15 (61)	19·8 (5)	19·3 (5)	19·70 ± 0·14 (38)	19·30 ± 0·15 (38)
15. Head index	95·6 (7)	96·7 (7)	95·33 ± 0·13 (64)	96·29 ± 0·13 (64)	92·0 (6)	93·8 (6)	92·18 ± 0·24 (43)	93·84 ± 0·20 (43)

Table 1 (*continued*)

Constant	Standard deviations				Bilateral correlations	
Sex	Male		Female			
Series	Hsiu Chiu Shan					
Side	R.	L.	R.	L.	Male	Female
1. Maximum length	14·49 ± 0·88	13·92 ± 0·85	16·53 ± 1·28	15·87 ± 1·23	0·96 ± 0·01	0·97 ± 0·01
2. Total length	13·80 ± 0·85	13·32 ± 0·82	15·67 ± 1·24	14·97 ± 1·19	0·96 ± 0·01	0·97 ± 0·01
3. Br. prox. epiphysis	2·13 ± 0·13	2·23 ± 0·14	2·82 ± 0·22	2·58 ± 0·21	0·96 ± 0·01	0·98 ± 0·01
4. Br. dist. epiphysis	2·67 ± 0·16	2·60 ± 0·16	3·99 ± 0·31	3·88 ± 0·30	0·94 ± 0·01	0·91 ± 0·02
5. Max. diam. mid. shaft	1·09 ± 0·07	1·09 ± 0·07	1·42 ± 0·11	1·35 ± 0·10	0·83 ± 0·03	0·78 ± 0·04
6. Min. diam. mid. shaft	1·24 ± 0·08	1·10 ± 0·07	1·07 ± 0·08	1·02 ± 0·08	0·51 ± 0·06	0·50 ± 0·08
7. Circum. mid. shaft	4·97 ± 0·30	5·20 ± 0·32	4·19 ± 0·32	4·24 ± 0·33	0·54 ± 0·06	0·52 ± 0·08
8. Min. circum. shaft	4·76 ± 0·27	4·38 ± 0·25	3·54 ± 0·26	3·67 ± 0·27	0·56 ± 0·06	0·56 ± 0·07
9. Sag. diam. head	2·18 ± 0·13	2·12 ± 0·13	2·06 ± 0·15	2·18 ± 0·16	0·89 ± 0·02	0·76 ± 0·04
10. Trans. diam. head	2·20 ± 0·13	2·02 ± 0·12	1·86 ± 0·14	1·88 ± 0·14	0·86 ± 0·02	0·85 ± 0·03
11. Circum. head	6·34 ± 0·38	6·38 ± 0·38	7·44 ± 0·54	7·05 ± 0·51	0·96 ± 0·01	0·96 ± 0·01
12. Angle of torsion	8°·45 ± 0·52	8°·05 ± 0·49	7°·65 ± 0·59	6°·45 ± 0·50	0·46 ± 0·07	0·46 ± 0·09
13. Index of diaphysis	3·40 ± 0·21	3·23 ± 0·20	2·26 ± 0·17	2·71 ± 0·21	0·46 ± 0·07	0·54 ± 0·08
14. Caliber index	1·77 ± 0·11	1·73 ± 0·11	1·26 ± 0·10	1·39 ± 0·10	0·66 ± 0·05	0·60 ± 0·07
15. Head index	1·57 ± 0·09	1·57 ± 0·09	2·31 ± 0·17	1·98 ± 0·14	0·42 ± 0·07	0·55 ± 0·07

Consistent bilateral differences between means are also found in the case of the four measurements of shape. For all four series the angle of torsion and caliber index have greater values on the right, and the diaphysial and head indices have greater values on the left. Differences having the same signs have generally been found for other series.

Standard deviations and bilateral correlations are given in Table 1 for the bones from Hsiu Chiu Shan. With the aid of these constants the statistical significance of the bilateral differences between the means can be tested.* The differences divided by their probable errors give the following values for the series in question, a negative sign indicating that the left mean is greater than the right:

No. of measurement	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
♂	10·6	7·7	3·7	7·6	14·4	1·4	3·7	4·1	24·9	9·4	11·4	0·7	−6·3	3·2	−6·9
♀	7·3	5·3	3·5	7·4	8·2	−0·3	3·4	4·4	7·5	4·9	4·0	1·0	−12·4	3·1	−7·9

The corresponding values of the ratios indicate a close agreement between the two sexes, and the majority provide clear evidence of asymmetry for both. It may be said, without regard to sex or race, that on the average the right humerus tends to be larger than the left in all respects, but that the side difference is much less marked for some transverse measurements of the shaft (nos. 6, 7 and 8), other than the maximum diameter of the section at the middle (5), than for the length of the bone and most measurements of its extremities. The torsion (12) tends to be greater on the right than on the left, and the same is true for the caliber index, or index of robustness (14). By far the most significant bilateral differences in shape, however, are for the index (13) expressing the minimum diameter at the middle of the shaft as a percentage of the maximum, which is greater on the left, and for the index (15) expressing the transverse as a percentage of the sagittal diameter of the head, which shows a difference of the same sign.

The significance of the bilateral differences between the standard deviations given in Table 1 for the longer of the series was estimated, as in the case of the means, by dividing each difference by its probable error.† For the fifteen characters, the highest of these ratios found for the male comparisons is 2·0, so all the differences must be considered quite insignificant. The highest ratio for the female comparisons is 4·0 (breadth of proximal epiphysis) and the next highest is 2·0. The former might be considered to indicate a significant difference if it were found by itself, but no importance can be attached to it considered as the extreme in a set of ratios. So far there appears to be no evidence of distinction between the variabilities of characters of the humerus on the right and left sides. For the eleven measurements of size, however, the standard deviations for the right side tend to be greater than those for the left. For these male constants the right value is the greater in seven cases, there is equality for one and the left is the greater in three cases: for the female constants the right is the greater in seven cases and the left is the greater in the remaining four. The tendency for the absolute variabilities of size measurements to be greater on the right accords with the fact that the right humerus is larger than the left, and

* The formula for the probable error of a difference is $0{\cdot}6745\sqrt{(\sigma_R^2+\sigma_L^2-2r_{RL}\sigma_R\sigma_L)}/\sqrt{n}$.

† The formula used for the probable error of the difference is $0{\cdot}6745\sqrt{(\sigma_R^2+\sigma_L^2-2r_{RL}^2\sigma_R\sigma_L)}/\sqrt{(2n)}$.

measures of relative variability (coefficients of variation) would doubtless make less distinction between the sides. A few markedly significant bilateral differences have been found for the absolute and relative variabilities of single bones of the cranium (Woo, 1931), but these criteria would not be expected to indicate distinction for series as short as those of the humeri from Hsiu Chiu Shan.

The bilateral correlations are given in the last two columns of Table 1. The average values for the male constants are 0·85 for the eleven size measurements and 0·50 for the four indices, the corresponding female means being 0·80 and 0·54. These may be compared with the means of 0·82 found for twenty-five absolute measurements of single bones of the cranium and 0·74 for twelve indices derived from them in the case of a long Egyptian series (Pearson & Woo, 1935). The correlations are high for the lengths, for all the absolute measurements of the epiphyses and for the maximum diameter at the middle of the shaft, but low—considered as bilateral correlations—for the minimum diameter of the same section and for the two circumferences of the shaft.

5. *Sexual comparisons.* All the male means in Table 1 are clearly greater than the corresponding female values. The following sex ratios for the absolute measurements (male mean/female mean) are found for the Hsiu Chiu Shan series of right bones:

No. of measurement	1	2	3	4	5	6	7	8	9	10	11
Sex ratio	1·089	1·094	1·103	1·100	1·120	1·259	1·181	1·142	1·172	1·202	1·130

The ratios are lower for the lengths (1 and 2) than for the other characters, but all are decidedly high for skeletal measurements. The average for the eleven characters is 1·145, and the average sex ratio for fifty-nine absolute measurements of the cranium given by the unpublished series from Hsiao T'un is 1·052. For all the series in the table the male means of the angle of torsion and of the three indices are greater than the corresponding female values, and for the Hsiu Chiu Shan all these differences are seen to be markedly significant except the two between the angles of torsion. Judging from the new Chinese series, the type of the male humerus has less torsion than the female, a less flattened section of the shaft, with greater girth relative to the length of the bone, and a head which approaches more nearly a circular form. Some of these relations do not accord with those found for other series. Table 3 gives the means of the Hsiu Chiu Shan and those of longer series of Lapp (Schreiner, 1931) and Norwegian (Wagner, 1927) bones. The three agree in showing a greater male than female mean for the caliber index, and all the differences may be supposed markedly significant. The index of the diaphysis is markedly greater for the Chinese male than for the female series; the Norwegian show a difference of the same sign which is much smaller but still probably significant, and the male and female means for Lapps are almost identical. Data for other races suggest that the diaphysial index is normally greater for males than for females, on the average. For the Chinese series there is also clear male dominance in the case of the head index, but the differences for Lapps and Norwegians are quite insignificant. Even less consistency is found for the angle of torsion. The male mean for right humeri of the Hsiu Chiu Shan series exceeds the female by 2°·4, which is 2·2 times the probable error of the difference, but for both Lapps (Δ/P.E. $\Delta = 3{\cdot}3$) and Norwegians (5·4) the female mean exceeds the male. It is possible that there are racial

distinctions in the sexual difference of humeral characters, but more extensive material would be required to establish this point.

No significant differences are found for the Hsiu Chiu Shan series between the corresponding male and female standard deviations, or between the bilateral correlations.

6. *Racial comparisons of mean measurements.* Metrical data relating to series of bones are far less extensive for the humerus than for the cranium, and considerably less numerous for the humerus than for the femur. The value of the available material is lessened appreciably owing to the fact that different definitions of measurements have been used by different workers, and also because constants are only given for sides, or sexes, or both, combined in the case of some of the series. As there are significant side and sexual differences, it is most undesirable that any such combinations should be made.

Table 2 gives mean measurements for the two Chinese series measured by Black (1925) and the two described for the first time in the present paper.* Black gives data for two other characters, but the definitions used do not accord with Martin's. Close agreement between the means is found in nearly all cases in spite of the fact that three of the series are very small. There is no suggestion of a significant difference for any one of the five characters.

In Table 3 comparison is made between the modern series from Nanking, one of Lapps described by Schreiner (1931) and one of Norwegians described by Wagner (1927). All the measurements recorded are involved in this case and all the series are adequate in length for both sexes. No significant differences are found between the lengths of the Chinese and Lapp series. In the case of the male means of the absolute measurements differences exceeding 3·5 times their probable errors are only found for the maximum diameter of the shaft at the middle (Δ/P.E. $\Delta = 6{\cdot}0$), for the minimum diameter of the shaft (4·7), and for the transverse diameter of the head (4·7). For the first two of these measurements the Lapp mean exceeds the Chinese, and for the third the Chinese is the greater. In the case of the female means of the absolute measurements the significant differences are for the minimum diameter of the shaft at the middle (15·1), the circumference of the shaft (4·8), the sagittal diameter of the head (3·9), and the transverse diameter of the head (5·2), the Lapp mean being the greater for all four. These relations are appreciably different for the two sexes and the measurements of shape show the same discordance. For both males (7·6) and females (11·7) the angle of torsion is greater for the Lapp bones. For males the difference for the index of the diaphysis is insignificant, and for females the Lapp mean exceeds the Chinese by an amount which is 23·0 times its probable error. The head index also show unexpected relations, the Chinese male mean being significantly greater than the Lapp (8·3), while the Lapp female mean is significantly greater than the Chinese (5·6).

Generalization is difficult in view of these results. One clear conclusion, however, is that the average types of the humerus for two racial populations may differ with marked significance in the case of some features when the total lengths are undifferentiated. Judging from the data for both sexes, it can be stated that the Lapp bone tends to have a more massive section of the mid-shaft and a greater angle of torsion than the Chinese, but no other clearly significant and consistent differences are found. In the comparison of the two series, there is marked disagreement between the male and female relations in the case of the index of the diaphysis and of the head index. It can be seen from Table 3 that the

* Measurements of other series of Chinese humeri have probably been given by Haberer (1892) and Kurz (1922), but I was unable to consult these sources.

Table 2. *Mean measurements of Chinese series of right humeri**

		1. Maximum length	4. Br. dist. epiphysis†	5. Max. diam. mid. shaft	6. Min. diam. mid. shaft	13. Index of diaphysis (6/5)
♂	Yang Shao: prehistoric	332·0 (2)	61·2 (4)	23·5 (4)	17·7 (4)	75·4 (4)
	Hsiao T'un: medieval	312·9 (7)	59·8 (6)	23·4 (11)	18·1 (11)	77·1 (11)
	North China: modern	311·7‡ (20)	60·2 (20)	22·7 (20)	17·2 (20)	75·7 (20)
	Hsiu Chiu Shan: modern	310·3 ± 1·2 (61)	59·4 ± 0·23 (64)	22·4 ± 0·09 (61)	17·1 ± 0·11 (61)	76·7 ± 0·29 (61)
♀	Yang Shao: prehistoric	284·2 (4)	55·5 (4)	20·4 (5)	14·4 (5)	70·7 (5)
	Hsiao T'un: medieval	286·4 (5)	54·3 (7)	20·0 (7)	13·9 (7)	68·9 (7)
	North China: modern	285·6 (3)	50·3 (3)	19·0 (3)	13·0 (3)	68·5 (3)
	Hsiu Chiu Shan: modern	284·9 ± 1·8 (38)	54·0 ± 0·44 (38)	20·0 ± 0·16 (38)	13·6 ± 0·12 (38)	68·3 ± 0·25 (38)

* The Yang Shao Tsun and north China series are Black's (1925) and the other two are described in the present paper.

† Black determined the maximum breadth of the distal epiphysis with small callipers and I followed Martin in measuring it on the board, but readings obtained in the two ways may be supposed fairly comparable.

‡ Stevenson (1929) gives a maximum length of 310·7 ± 1·2 for forty-eight male and right humeri in the collection of modern Chinese skeletons at the Peiping Union Medical College. This series almost certainly includes the series measured earlier by Black.

Table 3. *Mean measurements (with probable errors) of the modern Chinese and two other racial series of humeri*

Characters*	Male			Female		
	Chinese (Hsiu Chiu Shan)	Lapp	Norwegian	Chinese (Hsiu Chiu Shan)	Lapp	Norwegian
1. Maximum length (1)	310·3 ± 1·25 (61)	308·0 ± 0·85 (148)	337·3 ± 0·86 (162)	284·9 ± 1·81 (38)	284·8 ± 0·81 (131)	312·7 ± 0·97 (158)
2. Total length (2)	304·5 ± 1·20 (60)	302·5 ± 0·86 (146)	331·0 (157)	278·5 ± 1·76 (36)	280·5 ± 0·80 (127)	307·6 (148)
3. Br. prox. epiphysis (3)	48·0 ± 0·19 (59)	48·1 ± 0·20 (97)	51·9 ± 0·15 (147)	43·5 ± 0·32 (36)	43·3 ± 0·15 (99)	46·4 ± 0·16 (142)
4. Br. dist. epiphysis (4)	59·4 ± 0·23 (64)	60·5 ± 0·22 (103)	65·1 ± 0·19 (152)	54·0 ± 0·44 (38)	54·2 ± 0·20 (82)	57·7 ± 0·22 (149)
5. Max. diam. mid. shaft (5)	22·4 ± 0·09 (61)	23·2 ± 0·10 (141)	24·0 ± 0·08 (161)	20·0 ± 0·16 (38)	20·4 ± 0·08 (120)	21·1 ± 0·09 (158)
6. Min. diam. mid. shaft (6)	17·1 ± 0·11 (61)	17·8 ± 0·10 (141)	19·0 ± 0·07 (161)	13·6 ± 0·12 (38)	15·7 ± 0·07 (120)	16·2 ± 0·08 (158)
7. Circum. mid. shaft (7*a*)	67·8 ± 0·43 (61)	68·1 ± 0·28 (140)	72·0 ± 0·21 (160)	57·4 ± 0·46 (38)	59·9 ± 0·24 (120)	62·1 ± 0·23 (158)
8. Min. circum. shaft (7)	64·0 ± 0·38 (70)	64·4 ± 0·26 (141)	67·6 ± 0·20 (162)	56·0 ± 0·37 (42)	56·6 ± 0·22 (120)	57·9 ± 0·21 (158)
9. Sag. diam. head (10)	46·1 ± 0·18 (65)	45·5 ± 0·18 (115)	48·5 (151)	39·3 ± 0·21 (43)	40·3 ± 0·15 (96)	43·2 (145)
10. Trans. diam. head (9)	43·6 ± 0·19 (64)	42·3 ± 0·20 (94)	45·2 (135)	36·3 ± 0·19 (43)	37·6 ± 0·16 (84)	40·4 (121)
11. Circum. head (8)	137·6 ± 0·53 (65)	138·0 ± 0·64 (89)	147·7 ± 0·45 (116)	121·7 ± 0·77 (43)	122·8 ± 0·47 (79)	131·1 ± 0·55 (107)
12. Angle of torsion (18)	149°·3 ± 0·73 (61)	156°·1 ± 0·52 (125)	161°·1 ± 0·45 (149)	146°·9 ± 0·84 (38)	158°·6 ± 0·54 (113)	165°·0 ± 0·56 (140)
13. Index of diaphysis (6/5)	76·7 ± 0·29 (61)	77·0 ± 0·32 (141)	78·8 (161)	68·3 ± 0·25 (38)	77·1 ± 0·29 (120)	77·0 (158)
14. Caliber index (7/1)	20·7 ± 0·15 (61)	20·9 ± 0·08 (140)	20·1 (162)	19·7 ± 0·14 (38)	19·9 ± 0·08 (119)	18·5 (158)
15. Head index (9/10)	95·3 ± 0·13 (64)	93·1 ± 0·23 (93)	93·4 (135)	92·2 ± 0·24 (43)	94·2 ± 0·26 (84)	93·6 (121)

* The numbers preceding the measurements are those used in this paper, and the numbers in brackets following the measurements are those of Martin's list.

corresponding male and female means of these two characters differ insignificantly in the case of the Lapp series. In the case of the Norwegian, the sexual difference is insignificant for the head index and probably significant for the index of the diaphysis. In the case of the Chinese the differences are far larger and both are markedly significant. Two possible explanations of this situation may be suggested. The first is that the male and female Chinese series do not represent precisely the same population. The second is that the two measures of shape referred to may be influenced to some extent by conditions of life which were much more dissimilar for men and women in the Chinese than in the Lapp and Norwegian populations. The former appears to be the more plausible hypothesis, but more extensive data for racial series of humeri will be needed to clarify the position.

Probable errors are not provided for some of the Norwegian means quoted in Table 3, but it is clear that they exceed the corresponding Chinese and Lapp values with marked significance in the case of all the absolute measurements. The Norwegian bone is the largest in all respects. It also has the largest angle of torsion in the case of both sexes. Less significant differences are found for the three indices, but they still differentiate the Norwegian from both Chinese and Lapp means in one or more instances. Racial types of the humerus are evidently distinguished in shape as well as in size.

Table 4. *Frequencies of septal apertures for adult series of humeri (R. and L.)*

		Reported by	Total no.		No. with apertures		%	
			♂	♀	♂	♀	♂	♀
1	Prehistoric Chinese	Black	7	13	1	1	14·3 ± 8·9	7·7 ± 5·0
2	Medieval Chinese	Woo	21	15	2	2	9·5 ± 4·3	13·3 ± 5·9
3	Modern Chinese	Woo	162	94	14	10	8·6 ± 1·5	10·6 ± 2·1
4	Modern Chinese	P'an Ming-tzu	226	14	17	4	7·5 ± 1·2	28·6 ± 8·1
1–4	All Chinese	Black, etc.	416	136	34	17	8·2 ± 0·9	12·5 ± 1·9
5	Koreans	Akabori	233	20	16	3	6·9 ± 1·1	15·0 ± 5·4
6	Japanese	Akabori	462	250	45	69	9·7 ± 0·9	27·6 ± 1·9
7	Ainu	Akabori	108	74	15	14	13·9 ± 2·2	18·9 ± 3·0
8	'Whites' in U.S.A.	Hrdlička & Trotter	3213	1095	133	102	4·1 ± 0·2	9·3 ± 0·6
1–7	Orientals	Akabori, etc.	1219	480	110	103	9·0 ± 0·6	21·5 ± 1·3
9	American Negroes	Hrdlička & Trotter	704	252	67	66	9·5 ± 0·7	26·2 ± 1·9
10	Eskimos	Hrdlička	588	569	56	171	9·5 ± 0·8	30·1 ± 1·3
11	American Indians	Hrdlička	1665	1420	308	604	18·5 ± 0·6	42·5 ± 0·9
12	Ancient Egyptians	Hrdlička	264	309	87	176	33·0 ± 2·0	57·0 ± 1·9

7. *Anomalies.* The only anomaly of the humerus of anthropological interest for which extensive records are available is the opening in the bony septum that separates the coronoid from the olecranon fossa, known as a septal aperture. In a comprehensive memoir Hrdlička (1932) has collected data from the earlier literature relating to the frequency of occurrence of this condition in racial series of bones and added extensive observations of his own. He concludes that the anomaly is inherited and his figures show that it makes suggestive racial distinctions. For the same population the aperture occurs more frequently in left bones than in right, and also more frequently in females than in males. In view of these differences it is most desirable that data should be given for sides separately and for

sexes separately. Unfortunately the only records now available for the new Chinese series relate to right and left bones taken together for each sex considered separately.*

The figures are given in Table 4, together with others provided by Black (1925), P'an Ming-tzu (1935), Akabori (1934), Hrdlička (1932) and Trotter (1934). There are data for four Chinese series but the numbers of bones are small. For each sex considered separately no significant differences are found between the percentage frequencies. For all four series combined the female frequency exceeds the male but the difference is still insignificant. A lack of evidence of distinction is also found in the comparison of the pooled Chinese with the other three Oriental series in the table in the case of male bones. For female bones the percentages show one significant difference: the Japanese value exceeds the Chinese by an amount which is 5·6 times its probable error. These comparisons confirm the conclusion that the frequencies with which septal apertures are found tend to be practically constant for racial populations belonging to the same family of races. The data in the lowest section of Table 4 show that they make some marked distinctions between different families of races. The Oriental, American Negro and Eskimo groups show no significant differences for males and only one for females, but otherwise all the percentages differ with marked significance. The frequency of occurrence of the anomaly is aligned with such a character as skin colour which shows little variation within continental populations but some marked distinctions between such groups.

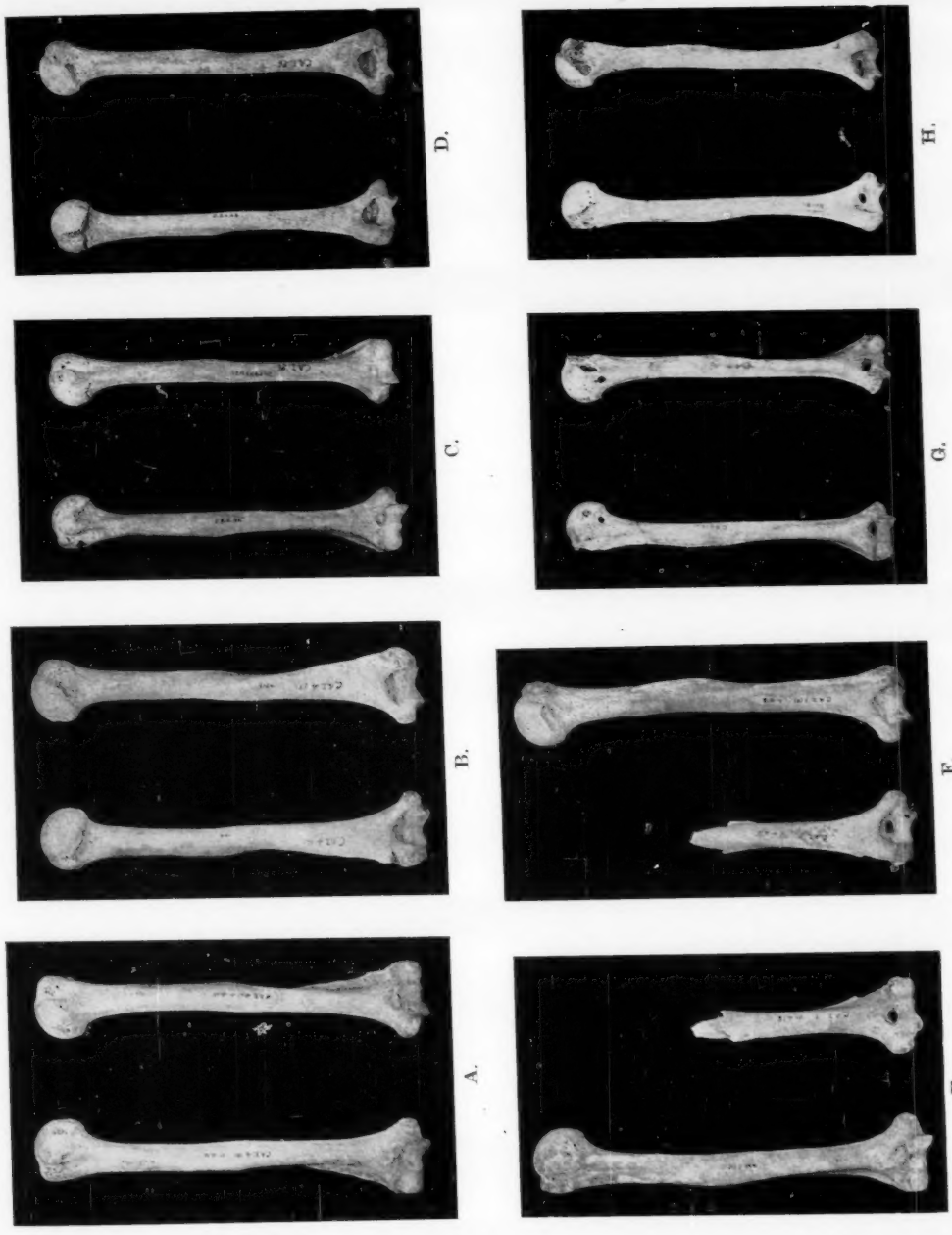
No example of the supracondyloid process was found in either of the Chinese series described in this paper. Black (1925) noted its absence from the prehistoric and modern Chinese humeri which he examined. As far as can be told from the available records, the anomaly is found more frequently in European than in any other populations (Terry, 1930).

* The sizes of the apertures show considerable variation from "pin-points" to holes having a maximum breadth of several mm. Photographs of some of the bones showing the largest apertures are reproduced in Plate I. Only one example of a double aperture was found, this being a female right humerus of the Hsiao T'un series.

EXPLANATION OF PLATE 1.

A. No. 4, ♂, *R* and *L*. Anterior view of typical bones.
B. No. 4, ♂, *R* and *L*. Posterior view of typical bones.
C. No. 26, ♀, *R* and *L*. Anterior view of typical bones.
D. No. 26, ♀, *R* and *L*. Posterior view of typical bones.
E. No. 1, ♂, *R* and *L*, anterior view. Septal aperture in left bone.
F. No. 1, ♂, *R* and *L*, posterior view. Septal aperture in left bone.
G. No. 16, ♀, *R* and *L*, anterior view. Septal aperture in both bones.
H. No. 16, ♀, *R* and *L*, posterior view. Septal aperture in both bones.

Woo: *Chinese humeri from Hsiao T'un* (all *c.* one-fifth natural size)



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VARIATIONS IN THE WEIGHTS OF HATCHED AND UNHATCHED DUCKS' EGGS

By J. M. RENDEL, *Department of Biometry, University College, London at Rothamsted Experimental Station, Harpenden*

Landauer (1941) has reviewed a great deal of evidence showing that hatchability of the domestic hen's egg depends in part upon its weight. Very large eggs hatch less frequently than medium-sized eggs. The effect of small weight on hatchability is more obscure; it seems probable that small eggs hatch less frequently than medium-sized eggs, but more frequently than large ones. As eggs of extreme weight are less likely to hatch than eggs of medium weight, we may expect to find that the eggs which hatch are less variable in weight than those which fail to do so, and that selection is acting against variation in egg weight. This paper gives an analysis of the variation in weight of the eggs of domestic ducks, and the effect of egg weight on hatchability.

During an experiment on some factors which might influence the weight of table ducklings (Rendel, 1941) some data were collected on the relationship between the weights of eggs which hatched, and the weights of eggs which failed to do so. Two breeds of duck were used in this experiment, a strain of Aylesbury duck which I have labelled N.P.I., and a strain labelled Allports. The two breeds have been treated separately except in the graph, where the figures from the two breeds have been added together.

The weights of the eggs laid by the two groups are given in Table 1. In the N.P.I. group birds labelled 71–82 are in their first laying season, and nos. 248–283 are in their second. In the Allport group nos. 408–416 are in their first season and nos. 208–332 are in their second.

All the eggs recorded here were laid between 26 February and 7 May 1939, a period of 10 weeks. Eggs were collected at the same time every day, and washed immediately. They were allowed an hour to dry before being weighed. As ducks lay their eggs in the morning fairly regularly, insufficient time will elapse between egg laying and weighing to allow an appreciable loss of weight by evaporation. After 1 week's incubation the eggs were examined by candling. All eggs which had not started to develop at all were removed and it was assumed that they had not been fertilized. The remainder were classed as fertile eggs and are those recorded here. It is probable that some eggs which had in fact started to develop, but which had died before the growth of any blood vessels, were classified as infertile. It is unlikely that this error was large or that it would be distributed so as to affect the results in any way. Abnormal eggs, such as double-yolked eggs, shell-less eggs, and very small eggs weighing less than 45 g. with gross deficiencies of yolk or white, were not incubated. There were very few of these.

The Allport and N.P.I. ducks differed considerably from each other. The N.P.I. is a large breed, the Allport is much smaller and, age for age, lays more eggs. The total number of eggs recorded here for N.P.I. ducks is larger than the number recorded for the Allport ducks because there was a higher proportion of young ducks in the N.P.I. group.

The cumulants of the weights of the eggs are given in Table 2. Apart from a slight difference in egg weight, the only major difference between Allport and N.P.I. is in the 4th cumulant, κ_4.

Table 1. *Egg weights*

A. *N.P.I. ducks. Weight of fertile eggs which hatched*

Wt. of eggs in g.	No. of dam																							Total
	71	72	73	74	75	76	77	78	79	80	81	82	248	252	257	260	261	266	276	279	280	282	283	
105	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1
104	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
103	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
102	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
101	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
100	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1
99	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
98	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
97	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
96	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
95	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
94	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
93	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	.	.	.	.	.	2
92	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1
91	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	2
90	.	.	1	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	3
89	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	2
88	.	.	3	.	.	.	.	.	.	.	.	1	.	.	1	.	.	1	.	.	.	.	.	6
87	.	.	4	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	6
86	.	.	.	.	.	.	.	.	1	.	.	.	.	.	2	.	.	.	.	.	.	1	.	4
85	2	.	2	.	.	.	1	.	.	.	1	.	.	.	.	1	.	.	.	.	1	.	.	8
84	2	.	1	.	.	.	.	1	.	.	1	1	.	.	1	.	.	.	.	.	.	1	.	8
83	.	.	2	1	1	1	.	2	.	.	3	1	.	1	1	.	1	.	.	1	.	.	.	15
82	.	.	2	.	1	.	.	.	.	.	1	1	.	1	.	.	.	1	1	1	.	1	.	10
81	1	.	2	.	1	.	1	.	3	.	4	.	.	3	1	.	2	1	.	.	.	.	.	19
80	.	.	.	.	3	.	5	.	1	.	2	1	1	.	.	.	1	2	1	2	.	.	2	21
79	1	.	3	1	2	1	1	2	.	.	.	2	.	1	.	.	.	5	2	2	.	1	3	27
78	3	.	5	.	1	3	7	1	1	.	2	1	1	2	1	2	2	.	1	1	1	.	1	36
77	3	.	1	1	2	2	5	.	.	.	1	.	2	1	.	1	4	1	1	2	.	1	1	29
76	3	.	4	1	3	2	6	.	1	.	1	1	.	1	.	1	4	1	1	1	4	.	3	38
75	2	.	1	.	5	3	6	2	2	.	1	1	1	.	.	4	.	.	3	1	3	.	2	37
74	8	1	4	.	2	2	4	.	2	.	.	.	.	1	.	1	1	1	1	2	4	2	3	39
73	6	2	.	2	1	1	1	1	2	.	.	.	.	.	.	2	1	2	2	2	.	1	2	28
72	5	3	2	.	1	.	3	1	1	1	1	.	2	.	.	.	2	2	2	2	1	.	3	32
71	3	.	3	.	.	1	1	2	4	6	1	.	1	.	.	3	2	.	2	3	.	3	.	35
70	.	4	2	.	1	2	.	1	4	4	.	.	1	.	.	3	1	6	1	3	1	6	3	43
69	.	3	.	.	1	.	.	.	1	6	.	.	3	1	.	2	1	2	3	2	.	2	.	27
68	1	1	1	1	2	2	.	.	2	9	.	.	3	.	.	2	.	1	2	.	2	4	1	34
67	.	3	.	.	.	1	.	1	.	5	.	.	5	.	.	.	1	5	1	.	.	1	.	23
66	.	4	.	.	.	1	.	.	.	5	.	.	2	.	.	2	1	3	1	1	.	.	1	21
65	.	2	.	.	.	6	.	.	3	9	.	.	1	.	.	.	.	.	1	.	3	1	1	27
64	.	2	.	.	.	4	.	.	.	1	.	.	2	.	.	.	.	.	.	.	3	.	2	14
63	.	.	.	.	.	2	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	1	6
62	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	3
61	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1
60	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1
59	.	1	.	.	.	.	.	.	.	.	.	.	2	.	.	.	.	.	.	.	.	.	.	3
58	.	.	.	3	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	5
57	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
56	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
55	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
54	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
53	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
52	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
Total	40	26	43	11	27	35	42	15	30	48	22	10	30	13	13	25	24	34	27	27	23	25	29	619

Table 1 (*cont.*)

B. *N.P.I. ducks. Weight of fertile eggs which did not hatch*

Wt. of eggs in g.	No. of dam																							Total
	71	72	73	74	75	76	77	78	79	80	81	82	248	252	257	260	261	266	276	279	280	282	283	
105	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
104	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
103	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
102	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
101	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
100	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
99	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1
98	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
97	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1
96	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3	.	.	.	.	.	.	.	.	3
95	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
94	.	.	.	.	.	.	.	.	.	.	.	.	.	.	5	.	.	.	.	.	.	.	.	5
93	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
92	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	2
91	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	.	.	.	.	.	2
90	.	.	.	.	.	.	.	.	.	.	1	1	.	1	2	.	.	.	1	.	.	.	.	6
89	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	2
88	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1
87	.	.	.	.	.	.	.	.	.	.	.	3	.	.	.	.	1	.	.	.	.	.	.	4
86	1	1	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	1	1	.	.	.	6
85	.	.	.	.	.	.	.	.	.	.	1	.	.	1	2	.	.	.	.	.	.	.	.	4
84	1	.	.	.	1	.	.	.	.	.	2	.	.	1	.	1	.	.	.	1	.	.	.	7
83	1	.	.	2	1	.	1	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	7
82	1	.	1	.	1	.	.	1	.	.	3	2	.	1	.	.	.	.	2	.	.	.	.	12
81	.	1	.	.	.	.	.	.	.	.	.	2	.	.	.	.	.	.	.	.	.	1	1	5
80	2	.	.	.	1	1	5	2	1	.	.	1	.	.	.	.	1	.	.	1	.	.	.	15
79	.	.	.	.	.	1	3	.	.	.	.	.	.	3	2	.	1	1	1	1	2	2	.	17
78	.	.	.	1	1	.	1	.	.	.	.	.	.	.	1	1	.	.	1	1	1	2	.	10
77	2	.	.	.	1	1	2	.	.	.	.	.	.	2	2	1	.	.	.	2	2	1	.	16
76	.	.	.	.	.	.	1	2	2	.	1	1	.	1	.	1	.	1	.	1	2	.	.	13
75	1	.	1	.	.	.	2	.	.	.	1	.	.	1	.	1	.	.	1	.	2	4	.	14
74	.	.	.	1	1	.	1	2	1	.	.	.	1	.	.	.	3	.	.	.	1	1	.	12
73	.	.	1	1	.	.	1	2	.	.	.	.	.	.	.	2	1	.	2	1	4	1	3	19
72	1	.	1	.	.	1	.	.	1	1	1	.	.	.	.	2	1	.	1	2	1	.	2	15
71	3	1	.	1	.	.	.	.	1	.	.	.	.	.	.	4	.	.	.	4	.	3	1	18
70	.	.	1	.	.	.	.	1	1	.	.	.	1	.	.	.	.	.	2	2	2	2	1	13
69	.	1	.	1	.	3	.	1	1	2	1	.	.	.	.	.	1	1	2	.	1	4	1	20
68	.	3	.	.	.	.	.	.	1	.	.	.	1	.	.	.	4	4	.	1	2	1	.	17
67	.	4	.	1	.	2	.	.	.	5	.	.	.	.	.	.	1	3	1	.	.	1	.	18
66	.	3	.	.	1	.	.	.	1	.	.	.	2	.	.	1	.	1	1	.	5	.	.	15
65	.	2	.	1	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	1	2	.	.	8
64	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	2	.	1	6
63	.	2	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	3	.	.	7
62	.	1	.	1	.	.	.	.	.	.	.	.	1	.	.	.	3	.	.	.	2	.	.	8
61	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	.	.	2
60	.	.	.	1	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	2
59	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
58	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
57	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
56	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
55	.	.	.	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
54	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
53	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
52	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
Total	13	21	5	17	8	12	17	11	11	10	12	13	8	12	22	17	17	12	16	19	35	23	10	341

Table 1 (*cont.*)

C. *Allport ducks. Weight of fertile eggs which hatched*

Wt. of eggs in g.	No. of dam																								Total
	408	409	410	411	412	413	414	415	416	208	209	210	211	213	215	216	219	221	235	238	258	329	331	332	
97	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1
96	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
95	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1
94	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
93	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1
92	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
91	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1
90	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	2
89	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3	.	.	.	.	.	3
88	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.	1	3
87	.	.	.	.	.	.	2	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	3
86	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	2	.	.	.	.	.	3
85	.	.	.	.	.	.	1	1	.	.	.	1	.	.	.	.	.	.	2	1	.	.	1	.	7
84	.	.	.	.	.	.	1	.	.	.	.	1	.	1	.	.	.	.	1	1	.	.	1	.	6
83	.	.	.	.	.	.	3	2	.	.	.	.	.	.	.	.	.	.	1	1	.	1	2	.	10
82	.	1	.	.	.	.	2	3	1	1	.	1	1	.	.	.	.	.	.	1	.	.	2	.	13
81	.	5	.	.	.	.	6	1	2	1	.	1	2	.	.	.	.	.	.	1	.	.	1	.	20
80	.	3	.	.	.	1	4	2	.	.	.	2	1	.	.	.	.	.	1	2	.	.	2	.	18
79	.	1	.	.	1	1	3	1	2	1	2	.	.	1	.	1	1	.	.	.	1	.	2	.	18
78	.	5	1	.	1	1	3	2	4	.	1	.	3	.	.	.	1	.	.	.	.	2	3	.	27
77	.	2	1	.	1	1	2	2	4	1	.	.	2	1	1	.	2	.	.	.	.	1	1	1	23
76	.	1	.	.	1	1	5	1	2	4	3	.	4	.	.	1	2	.	.	1	.	.	.	1	27
75	2	3	.	1	3	2	.	5	.	3	3	.	4	2	.	1	2	1	.	.	2	1	1	.	36
74	.	2	2	.	1	1	6	3	1	2	.	.	.	.	1	.	2	1	.	1	.	4	2	2	31
73	1	2	3	1	3	3	1	2	1	1	1	.	2	1	.	2	3	.	.	1	2	2	.	2	34
72	1	1	.	2	.	4	1	1	2	5	1	.	1	2	.	3	1	2	.	4	3	1	.	6	41
71	2	.	.	.	.	3	.	2	3	5	1	.	2	.	.	2	2	1	.	2	4	2	.	3	34
70	.	.	1	4	4	2	1	2	1	2	1	.	2	1	.	6	1	2	.	2	4	.	1	.	37
69	3	.	1	.	1	1	2	.	.	1	.	.	3	1	1	4	.	4	.	.	1	.	.	.	23
68	1	.	.	3	2	4	.	.	1	1	.	.	.	2	1	6	.	6	.	.	2	1	.	3	33
67	4	.	.	.	2	.	.	.	.	.	2	.	.	.	.	7	.	2	.	.	2	4	.	4	27
66	5	.	1	1	1	.	.	.	.	.	2	.	.	2	3	1	.	3	.	1	1	.	.	1	22
65	2	.	.	.	3	6	.	.	.	.	2	.	1	.	3	2	.	1	.	.	.	1	1	2	24
64	2	.	1	.	4	1	.	.	.	.	.	.	.	.	5	4	.	1	.	.	.	1	.	.	19
63	2	.	2	1	1	.	.	.	.	.	2	.	.	.	3	.	.	1	.	.	.	6	.	.	18
62	1	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4	.	.	8
61	.	.	2	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	.	.	4
60	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	3
59	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
58	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1
57	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
Total	26	26	17	14	31	32	43	30	24	28	22	8	29	14	19	40	17	25	14	20	23	34	21	26	583

Table 1 (*cont.*)

D. *Allport ducks. Weight of fertile eggs which did not hatch*

Wt. of eggs in g.	No. of dam																								Total
	408	409	410	411	412	413	414	415	416	208	209	210	211	213	215	216	219	221	235	238	258	329	331	332	
97	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
96	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
95	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	.	.	2
94	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	.	.	2
93	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
92	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
91	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4	.	.	.	.	.	4
90	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	2
89	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
88	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1
87	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	3
86	.	.	.	.	.	.	1	1	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	4
85	.	.	.	.	.	.	.	.	.	.	.	.	.	3	.	.	.	.	1	.	.	.	1	.	5
84	.	2	.	.	.	.	2	1	.	.	.	.	.	3	.	.	.	.	.	1	.	.	.	.	9
83	.	1	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	1	.	6
82	.	2	.	.	.	3	2	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	2	.	10
81	.	2	.	.	.	2	2	.	.	.	.	.	2	1	.	.	.	.	.	1	.	.	1	.	11
80	.	2	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	1	.	.	.	4	.	10
79	.	3	.	1	.	1	2	.	1	1	1	1	.	1	.	.	.	.	.	1	.	1	1	.	15
78	.	4	.	1	.	.	.	.	3	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	10
77	.	4	2	.	1	1	2	.	2	1	.	.	.	.	.	.	.	.	.	.	1	1	.	.	15
76	.	.	.	2	2	4	.	1	1	1	1	.	1	3	.	.	.	.	.	1	.	3	1	.	21
75	.	4	1	2	1	.	.	2	2	.	.	.	2	1	.	.	.	.	.	3	.	1	.	2	21
74	.	1	.	.	.	2	1	2	1	3	.	.	1	1	.	.	.	.	.	1	.	1	2	1	17
73	1	.	.	1	2	1	1	1	.	1	.	.	.	2	.	2	2	.	.	.	1	1	1	1	17
72	.	.	2	2	1	.	.	1	.	2	2	.	.	2	.	.	.	.	.	1	.	2	.	1	16
71	6	.	3	3	4	1	.	1	.	1	2	.	.	.	.	1	1	.	.	2	2	.	.	.	27
70	4	.	.	.	1	2	.	2	1	.	1	.	.	.	1	.	1	.	.	.	.	3	1	2	19
69	5	.	.	1	1	.	.	.	.	.	.	.	1	.	.	1	.	1	.	.	1	1	1	1	14
68	3	.	1	1	2	1	.	.	.	1	.	.	.	1	.	5	.	1	.	1	1	.	.	2	20
67	1	1	.	3	1	2	.	.	.	1	.	.	1	.	.	2	.	2	.	.	.	.	.	.	14
66	1	.	2	.	1	2	.	.	1	1	.	.	.	.	1	.	.	1	.	.	1	.	.	.	11
65	1	.	.	2	.	.	.	.	.	.	.	.	.	.	.	1	.	2	.	.	.	1	.	1	8
64	2	.	2	1	2	1	.	.	.	.	.	.	.	.	.	2	.	.	.	.	1	1	.	.	12
63	1	.	.	2	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	1	.	.	6
62	1	.	1	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	.	.	4
61	2	.	1	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	1	.	.	6
60	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3	.	.	3
59	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
58	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1
57	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
Total	28	26	16	22	19	25	17	14	12	13	9	3	9	18	3	17	4	8	15	12	8	22	16	11	347

As measured by g_2 or κ_4/κ_3, the value of this cumulant is highly significant for the hatched N.P.I. eggs, but not for the unhatched. In the Allport group neither is significant. There is a considerable difference between hatched and unhatched eggs in both groups. There is a

Table 2

	N.P.I.			Allport		
	Hatched	Unhatched	Diff.	Hatched	Unhatched	Diff.
No. of eggs	619	341		583	347	
Mean weight in g.: κ_1	73·78 ± 0·27	74·17 ± 0·45	0·39 ± 0·35	72·87 ± 0·27	73·66 ± 0·38	0·79 ± 0·32
κ_2	43·87 ± 2·50	69·06 ± 5·30	25·19 ± 3·64	41·32 ± 1·71	49·76 ± 2·67	8·44 ± 2·14
κ_3	133·10	247·70	—	113·50	145·41	—
κ_4	1570·96	533·95	—	172·21	153·76	—
g_1	0·4581 ± 0·0980	0·4316 ± 0·1217	—	0·4273 ± 0·1010	0·4143 ± 0·1308	—
g_2	0·8164 ± 0·1960	0·1120 ± 0·2427	—	0·1001 ± 0·2025	0·0621 ± 0·2612	—

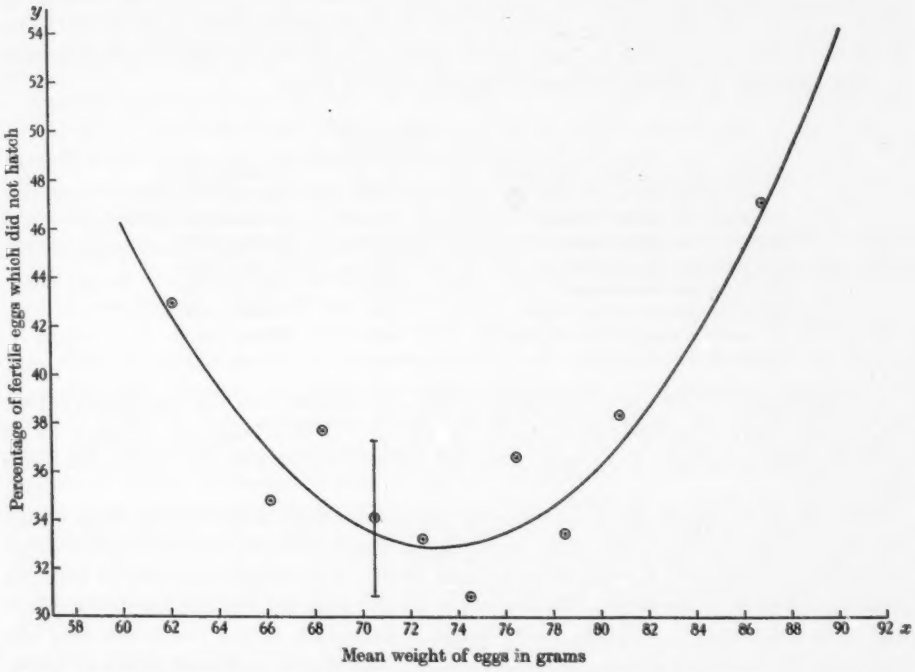

Fig. 1. Combined data for N.P.I. and Allport ducks. The curve is the parabola
$y = 441{\cdot}4 - 11{\cdot}147x + 0{\cdot}07604x^2$.

tendency for the unhatched eggs to be heavier than the hatched and the variance of the former is very much the greater. The distributions of both hatched and unhatched are about equally skew, but the 4th cumulant, κ_4, is greater in the hatched than the unhatched.

These figures show clearly that the further the weight of an egg is from the mean weight, the smaller is its chance of hatching. Egg weight has been plotted against the percentage of eggs which did not hatch in Fig. 1, and a curve fitted. The standard error of the point with the smallest standard error is shown on the graph as a vertical line. The curve shows to what extent egg weight influences hatchability. Eggs with weights outside the range of 70–76 g. lose rapidly in hatchability with each gram, that their weight moves further from the mean. At 67 and 80 g. the loss is approximately 1 % per g., and at 62 and 85 g. the loss is $1\frac{2}{3}$ % per g., with a total loss of 11 % over eggs of mean weight, whereas at 90 g. the rate of loss is $2\frac{1}{2}$ % per g., with a total loss of 21 % over eggs of mean weight.

In breeds of domestic ducks there has been strong selection in favour of a high egg weight. The results given here suggest that it would be advantageous for birds in which a large number of offspring is the first consideration to lay eggs which do not differ widely from the optimum weight for hatchability. It would be interesting to know whether breeds which have not been selected for egg weight by breeders had a variation in egg weight much less than the variation shown here. One would expect, for example, that wild breeds would give a much smaller variance.

Although there are no reliable figures on egg weights, since the time from laying is unknown, there are data in the literature on measurements of length and breadth of eggs of wild birds. From these data it is possible to estimate the amount of variation in egg volume (Appendix 1). The coefficient of variation of various species is given below:

	Coeff. of var. of egg vol.	
English sparrow (*Passer domesticus*)	10·12	Pearson (1901)
Common tern (*Sterna hirundo*)	8·69	Rowan *et al.* (1919)
Common tern (*Sterna hirundo*)	7·75	Watson *et al.* (1923)
Mallard (*Anas platyrhyncha*)	7·18	Fisher (1935)
Common wren (*Troglodytes t.*)	8·13	Fisher (1935)
Greylag goose (*Anser anser*)	10·60	Fisher (1935)
Green woodpecker (*Picus viridis*)	12·9	Fisher (1935)
White-tailed eagle (*Haliaetus albicilla*)	14·2	Fisher (1935)
Cuckoo (*Cuculus canorus*)	14·4	Fisher (1935)

The coefficient of variation of the weights of all eggs considered in this experiment is 9·47. This is well within the limits of variation shown by wild birds' eggs as measured by an estimate of their volume. It is higher than the value found for the wild ducks, this value being 7·18.

There are many factors which influence the egg weight of the duck. Though ducks do lay long uninterrupted series of eggs, they usually lay in clutches varying from two or three eggs to twelve or fifteen with a pause between each clutch. Egg weight is correlated with the position of the egg in the clutch. The first egg is heavier than the last and the intermediate eggs have intermediate weights. Nevertheless, hatchability is not correlated with the position of the egg in the clutch (Rendel, 1941). The time of year is another factor involved. Egg weight rises as the season advances. Then there are certainly some genetic factors. It is of interest therefore to find out whether variation in egg weight due to the differences between individual ducks or variation due to factors which influence the fluctuations of egg weight round the mean of each individual duck, differ significantly for eggs which hatch and eggs which fail to hatch. The former type of variation will be largely genetic, the latter will include many environmental factors, though it is possible that the range of the weights

of eggs laid by a duck is partly determined genetically. An analysis of variance of all good and bad eggs is shown below. The notation is that of Appendix 2.

	Degrees of freedom	Sum of squares	Mean square	
Good eggs: Within progenies	1155	27121	23·481	$\tau^2 = 23{\cdot}481$, variance 0·9547
Between progenies	46	24037	522·54	$\sigma^2 = 19{\cdot}570$, variance 3·9762
Total	1201	51158	42·596	
Bad eggs: Within progenies	641	18459	28·797	$\tau^2 = 28{\cdot}797$, variance 2·5874
Between progenies	46	22237	483·41	$\sigma^2 = 31{\cdot}193$, variance 12·2207
Total	687	40696	59·237	

τ^2 measures the variation which is considered to be largely environmental and σ^2, calculated from the mean square between progenies, measures the variation due to differences between individual ducks, which are considered to be of genetic origin (Appendix 2).

The difference between τ^2 for good and bad eggs is $5{\cdot}316 \pm 1{\cdot}882$ and the difference between σ^2 is $11{\cdot}626 \pm 4{\cdot}025$. Both differences are nearly three times their standard error. We may say therefore that there is selection against ducks which habitually lay heavy or light eggs. This means that there will be selection against certain genotypes and that selection will operate so as to reduce the variation between the mean egg weights of individual ducks in subsequent generations. There is also selection against ducks which lay eggs with a very wide range of egg weights. It does not necessarily follow that such selection will have any effect on future generations, though it may do so.

In the light of this result we may review some other similar results. Crampton (1904) found that the pupae of *Philosamia cynthia* which died were significantly more variable than the survivors as regards seven measurements. Weldon (1901) in *Clausilia laminata* and Di Cesnola (1907) in *Helix arbustorum* found that the shells of young individuals varied more than the shells of old ones. In such cases it was not clear how far the variation was due to nature or nurture. It is possible that the individuals differing most from the mean had been exposed to more extreme conditions and that these conditions brought about the greater variability as well as the higher mortality. It has been shown, for example, that the weights of heart, liver and kidneys when diseased are more variable than the weights of these organs when healthy, and the correlation between the weights of these organs is less when they are diseased (Greenwood, 1904). This increase in variability and decrease in correlation is thought to be due to reaction of the body to disease. If this is the case, death of the more variable individuals has no selective influence. In the case of the ducks' eggs it has been possible to eliminate the day-to-day variation in egg weight and therefore a good deal of the variation due to nurture. We can at least say that there is some selection against ducks which habitually lay very large or very small eggs.

SUMMARY

A description of the variation of the weights of eggs laid by forty-seven ducks is given. Eggs which hatch are compared to eggs which do not hatch. Variation in egg volume of some wild species of birds is compared with variations in egg weight of the domestic duck. It is concluded that selection will tend to reduce variation in egg weight.

I should like to thank Dr Helen Spurway for preparing the graph, and Prof. J. B. S. Haldane for his help in the mathematical treatment of the results and for contributing the two appendices.

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APPENDIX 1. THE COEFFICIENT OF VARIATION OF EGG VOLUME

BY J. B. S. HALDANE, F.R.S.

A number of workers have measured the length L and breadth B of birds' eggs and found them to be more or less normally distributed. All authors give the means $\bar{L}$ and $\bar{B}$. Pearson and his colleagues give the mean value $\bar{I}$ of the index $I = B/L$, which is also nearly normally distributed, and the coefficients of variation l, b and i, of L, B and I. Fisher gives the variances λ and β of the length and breadth, and their covariance κ.

It is required to find the coefficient of variation v of the volume V, supposing that $V = kLB^2$, where k is a constant. That is to say the eggs are all supposed to be of the same shape apart from changes of scale in L and B. We also suppose L and B to be normally correlated with coefficient ρ. Let $u^2 = 2\rho bl$.

Then $L = \bar{L}(1+lx)$, and $B = \bar{B}(1+by)$, where x and y are correlated reduced normal variates (with zero mean and unit variance). The means of odd powers and products of x and y vanish, and

$$\overline{x^2} = \overline{y^2} = 1, \quad \overline{xy} = \rho, \quad \overline{x^4} = \overline{y^4} = 3, \quad \overline{x^3y} = \overline{xy^3} = 3\rho, \quad \overline{x^2y^2} = 1+2\rho^2, \quad \overline{x^4y^2} = 3(1+4\rho^2).$$

$$V = k\bar{L}(\bar{B})^2(1+lx)(1+by)^2.$$

Hence
$$\bar{V} = k\bar{L}(\bar{B})^2(1+2lb\overline{xy}+b^2\overline{y^2}) = k\bar{L}(\bar{B})^2(1+b^2+u^2),$$

$$\overline{V^2} = \bar{V}^2(1+v^2) = k^2(\bar{L})^2(\bar{B})^4[1+l^2+4u^2+6b^2+3(2l^2b^2+u^4+4b^2u^2+b^4)+3b^2(l^2b^2+u^4)].$$

So
$$v^2 = l^2+2u^2+4b^2+2(2l^2b^2-l^2u^2-u^4-5u^2b^2-3b^4)+\ldots.$$

Since l, b, and u are of the order of 0·05, we may take

$$v = \sqrt{(l^2+2u^2+4b^2)}, \tag{1}$$

with an error of the order of 4 %, which is generally negligible.

Since $\lambda = (\bar{L})^2 l^2, \quad \beta = (\bar{B})^2 b^2, \quad \kappa = \rho \bar{L}\bar{B} l b = \tfrac{1}{2}\bar{L}\bar{B}u^2,$

$$v = \sqrt{\left(\frac{\lambda}{(\bar{L})^2} + \frac{4\kappa}{\bar{L}\bar{B}} + \frac{4\beta}{(\bar{B})^2}\right)}. \tag{2}$$

If i is given, but not κ, we proceed as follows:

$$I = \frac{\bar{B}}{\bar{L}}(1+by)(1+lx)^{-1}.$$

This may be expanded in a series, and the moments calculated. Theoretically they are all infinite, since if L is normally distributed it can be zero. However, the series for the moments are asymptotic expansions, and the first few terms give good approximations:

$$\bar{I} = \frac{\bar{B}}{\bar{L}}\left[1 + (l-\rho b)\sum_{r=1}^{\infty}\frac{l^{2r-1}(2r!)}{2^r r!}\right] = \frac{\bar{B}}{\bar{L}}[1 + (l^2 - \tfrac{1}{2}u^2)(1 + 3l^2 + 15l^3 + \ldots)],$$

$$\overline{I^2} = \left(\frac{\bar{B}}{\bar{L}}\right)^2 + \bar{I}^2 i^2 = \left(\frac{\bar{B}}{\bar{L}}\right)^2 [1 + 3l^2 - 2u^2 + b^2 + 3(5l^4 - 4l^2u^2 + 3l^2b^2 + 3u^4) + \ldots],$$

$$i^2 = l^2 + b^2 - u^2 + \text{etc.},$$

and

$$v = \sqrt{(3l^2 + 6b^2 - 2i^2)}, \tag{3}$$

again with an error of the order of about 4 %.

APPENDIX 2. INTERPRETATION OF THE GREATER VARIANCE OF THE UNHATCHED EGGS

BY J. B. S. HALDANE, F.R.S.

First let us consider those eggs which hatched. Let the weight of such an egg be $W + w$, where W is the true mean, or the mean of a very large sample. Let $w = x\sigma + y\tau$, where x and y are uncorrelated reduced normal variables, i.e. $\bar{x} = \bar{y} = 0$, $\overline{x^2} = \overline{y^2} = 1$; and let x be constant for any given duck. That is, σx represents the deviation from mean egg weight due to a given duck's make-up, and τy the deviation of the egg weight from $W + \sigma x$ during the duck's life. We wish to estimate σ and τ, and to know whether they differ significantly from σ' and τ', the corresponding quantities, for unhatched eggs.

Let there be n ducks laying N good eggs in all. Let the rth duck lay k_r good eggs, so that $N = \sum_{r=1}^{n} k_r$.

Then the total sum of squares of deviations

$$\Sigma(w - \bar{w})^2 = \left(\frac{N^2 - \Sigma k_r^2}{N}\right)\sigma^2 + (N-1)\tau^2,$$

and the corresponding variance (mean square) is $\dfrac{N^2 - \Sigma k_r^2}{N(N-1)}\sigma^2 + \tau^2$, or $\left[1 + \dfrac{N - \Sigma k_r^2}{N(N-1)}\right]\sigma^2 + \tau^2$, which is nearly $\sigma^2 + \tau^2$. The sum of squares of deviations within progenies of individual ducks $\Sigma(w - \bar{w}_r)^2 = (N-n)\tau^2$, and the corresponding mean square is τ^2. Subtracting the two sums of squares, the sum of squares of deviations between progenies is

$$(\Sigma \bar{w}_r - \bar{w})^2 = \frac{N^2 - \Sigma k^2}{N}\sigma^2 + (n-1)\tau^2,$$

and the mean square is

$$\frac{N^2 - \Sigma k^2}{N(n-1)}\sigma^2 + \tau^2.$$

We have therefore:

	Degrees of freedom	Sum of squares	Mean square
Within progenies	$N-n$	$(N-n)\tau^2$	τ^2
Between progenies	$n-1$	$\frac{N^2-\Sigma k^2}{N}\sigma^2+(n-1)\tau^2$	$\frac{N^2-\Sigma k^2}{N(n-1)}\sigma^2+\tau^2$
Total	$N-1$	$\frac{N^2-\Sigma k^2}{N}\sigma^2+(N-1)\tau^2$	$\frac{N^2-\Sigma k^2}{N(N-1)}\sigma^2+\tau^2$

The sum of squares of deviations between means of progenies, and the total, are readily found, and the variance τ^2 within progenies is found by subtraction. σ^2 is then readily found.

THE PROBABILITY INTEGRAL FOR TWO VARIABLES

By C. NICHOLSON, M.C., M.A., M.D.

1. Introduction

A geometrical approach to problems connected with the normal bivariate surface (Nicholson, 1941) suggested a method for the direct integration of the surface which could be used to calculate a table which might be simpler in use than the present table of d/N calculated by Everitt (1912), Lee (1915), Lee (1927), and Elderton, Moul, Fieller, Pretorius & Church (1930) and republished in *Tables for Statisticians and Biometricians*, Part II. The results of this inquiry are here presented, and I must acknowledge the assistance which I have received in the preparation of the paper from Prof. E. S. Pearson and Mr N. L. Johnson.

A brief recapitulation of the relevant results from the earlier paper will first be given. Fig. 1 shows diagrammatically an elliptic contour of the normal correlation surface, for

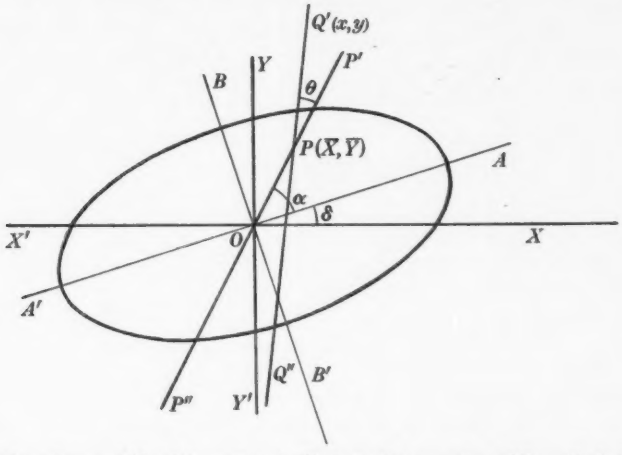

Fig. 1. Projection of normal bivariate surface to illustrate the geometry of the ratio between variables.

which the mean is at O, and the standard deviations and correlation coefficient of the variables are σ_x, σ_y, and r respectively. The principal axes make angles of δ and $\frac{1}{2}\pi+\delta$ with OX, and the standard deviations in these principal directions are a and b. If P is a point $(\bar{X}, \bar{Y})$, then the ratio of OP to the standard deviation of the plane section of the normal surface through OP is

$$p = \frac{1}{\sqrt{(1-r^2)}}\sqrt{\left\{\frac{\bar{X}^2}{\sigma_x^2} - \frac{2r\bar{X}\bar{Y}}{\sigma_x\sigma_y} + \frac{\bar{Y}^2}{\sigma_y^2}\right\}} \tag{1}$$

OP makes an angle α with the major axis of the ellipse. The earlier paper then considered the distribution of the ratio

$$v = \frac{y-\bar{Y}}{x-\bar{X}} = \tan(\alpha+\theta+\delta), \tag{2}$$

which is constant along $Q'PQ''$. The frequency distribution of v may clearly be derived

from that of θ, and it was shown that the distribution of θ could most simply be obtained from that of an angle ϕ given by

$$\phi = \tan^{-1}\{(a/b)\tan(\alpha+\theta)\} - \tan^{-1}\{(a/b)\tan\alpha\}. \tag{3}$$

Geometrically ϕ is the angle between lines corresponding to $P'OP''$ and $Q'PQ''$ after a transformation which substitutes for the correlation surface with its elliptical contours an equivalent system with circular contours having a common standard deviation for every section of $\sqrt{\left(\frac{a^2+b^2}{2}\right)}$, that is, $\sqrt{\left(\frac{\sigma_x^2+\sigma_y^2}{2}\right)}$. As a consequence of this transformation it follows that the cumulative frequency of θ and therefore of $\tan^{-1}v-(\alpha+\delta)$ is that of ϕ, or, as was proved,

$$P\{0\leqslant\theta\leqslant\Theta\} = \int_0^{\Phi}\frac{e^{-\frac{1}{2}p^2}}{\pi}\left\{1+p\cos\phi e^{\frac{1}{2}(p\cos\phi)^2}\int_0^{p\cos\phi}e^{-\frac{1}{2}x^2}dx\right\}d\phi, \tag{4}$$

where p and Φ are given in terms of Θ and the constants of the surface by equations (1) and (3). It was further shown that (4) may be expanded into a form

$$\phi/\pi + 2V(p,\phi) \tag{5}$$

where
$$V(p,\phi) = \frac{\sin\phi}{2\pi}\{A_0\cos\phi + A_1\cos^3\phi + A_2\cos^5\phi + \ldots\}, \tag{6}$$

and
$$A_0 = \int_0^p pe^{-\frac{1}{2}p^2}dp, \tag{7}$$

$$A_1 = \frac{1}{1\cdot3}\int_0^p p^3e^{-\frac{1}{2}p^2}dp, \tag{7a}$$

.....

$$A_n = \frac{1}{1.3.5\ldots(2n+1)}\int_0^p p^{(2n+1)}e^{\frac{1}{2}-p^2}dp. \tag{7b}$$

Now if we refer to Fig. 3 and suppose that the lines $P'OP''$, and $Q'PQ''$ of Fig. 1 have been transformed into $P'OP''$, and $H'PH''$, the cumulative frequency is the content of the double sector between the planes $H'PH''$ and $P'PP''$, and this is equal to the content of the double sector between the planes Y_hOY_h' and $P'OP''$, that is to ϕ/π, together with twice the content above the triangle OPH, so that $V(p,\phi)$ is the content of this triangle.

2. The standardized surface

We may now apply these conclusions to the special case of the standardized normal surface

$$z = \frac{N}{2\pi\sqrt{(1-r^2)}}\exp\left[-\frac{1}{2}\frac{x^2-2rxy+y^2}{1-r^2}\right],$$

which, using the nomenclature of *Tables for Statisticians and Biometricians*, is divided into four parts, a, b, c, and d by the two planes $x=h$, and $y=k$ parallel to the co-ordinate axes and intersecting in the ordinate at h, k. If we further subdivide this surface by a plane $P'OP''$ $(x/h=y/k)$ through the ordinates at the origin and at h, k we have the position in Fig. 2, and it will be appreciated that the content between the planes $P'PP''$ and $H'PH''$, as also the content between the planes $P'PP''$ and $K'PK''$ may be calculated separately by means of (4) (with the assumption that both deviations are positive), and that the sum of these contents is

$$\phi_h/\pi + 2V(p,\phi_h) + \phi_k/\pi + 2V(p,\phi_k) = (a+d)/N. \tag{8}$$

To do this we must, as before, refer the distribution of Fig. 2 to the equivalent symmetrical distribution obtained by transformation and illustrated in Fig. 3. Here

$$p = \sqrt{\left\{\frac{h^2 - 2rhk + k^2}{1-r^2}\right\}}, \tag{1a}$$

and, since $\sigma_x = \sigma_y$, the angle which the major and minor axes make with the primary co-ordinate axes is $\delta = \frac{1}{4}\pi$, so that

$$\alpha = \tan^{-1}(k/h) - \tfrac{1}{4}\pi = \tan^{-1}\{(k-h)/(k+h)\}. \tag{9}$$

The standard deviations on the major and minor axes of the ellipse are respectively $\sqrt{(1+r)}$ and $\sqrt{(1-r)}$. The angular deviation θ_h ($P'PH'$ in Fig. 2) is $\frac{1}{4}\pi - \alpha$, so that from (3)

$$\phi_h = \tan^{-1}\left\{\frac{\sqrt{(1+r)}}{\sqrt{(1-r)}}\right\} - \tan^{-1}\left\{\frac{(k-h)\sqrt{(1+r)}}{(k+h)\sqrt{(1-r)}}\right\} = \tan^{-1}\left\{\frac{h\sqrt{(1-r^2)}}{k-rh}\right\}. \tag{10}$$

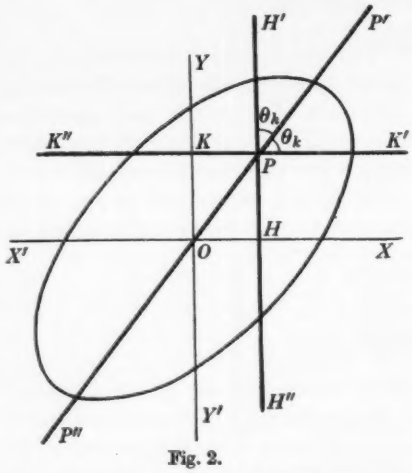

Fig. 2.

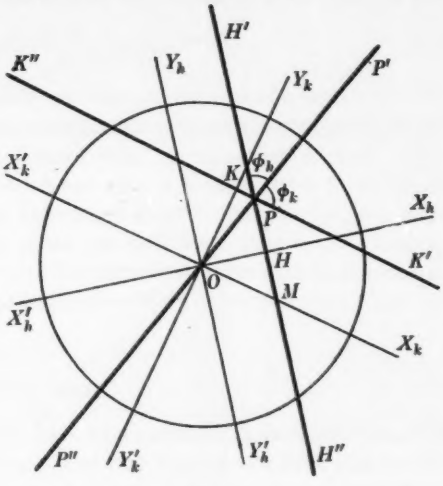

Fig. 3.

Fig. 2. Projection of the standardized normal surface $r = 0{\cdot}60$.

Fig. 3. Projection of the symmetrical surface equivalent to that illustrated in Fig. 2 to show the relationship between θ and ϕ.

Similarly, $\theta_k = \frac{1}{4}\pi + \alpha$, and

$$\phi_k = \tan^{-1}\left\{\frac{\sqrt{(1+r)}}{\sqrt{(1-r)}}\right\} + \tan^{-1}\left\{\frac{(k-h)\sqrt{(1+r)}}{(k+h)\sqrt{(1-r)}}\right\} = \tan^{-1}\left\{\frac{k\sqrt{(1-r^2)}}{h-rk}\right\}. \tag{10a}$$

It will be seen that for any value of r, $\phi_h + \phi_k$ is constant at all points in the distribution depending only on r; in fact

$$\phi_h + \phi_k = \kappa = 2\tan^{-1}\left\{\frac{\sqrt{(1+r)}}{\sqrt{(1-r)}}\right\} = \pi - \cos^{-1} r. \tag{11}$$

Also if we consider the triangle OPH in Fig. 3 we have the following identities:

$$\angle OPH = \phi_h = \tan^{-1}\left\{\frac{h\sqrt{(1-r^2)}}{k-rh}\right\}, \qquad OH = h,$$

$$OP = p = \sqrt{\left\{\frac{h^2 - 2rhk + k^2}{1-r^2}\right\}}, \qquad PH = q_h = PM - HM = \frac{k-rh}{\sqrt{(1-r^2)}}.$$

This is the position reached by Sheppard (1900) by another line of reasoning, and he goes on to discuss the calculation of a table of V and its application to practical work. Almost contemporaneously Pearson (1901) published a method for integrating the surface as a polynomial in r with coefficients which are functions of h and k, the tetrachoric functions of Everitt (1910). This method was devised primarily for the calculation of the correlation coefficient for a fourfold table which was supposedly normal in distribution, but the polynomial converges so slowly for high values of r that it was never very satisfactory. In its place a table of d/N was calculated for high positive values of r (Everitt, 1912) and for high negative values (Lee, 1915). In Fig. 3 since

$$PH = \frac{k - rh}{\sqrt{(1-r^2)}}$$

for all values of h, it will be seen that geometrically

$$d/N = \frac{1}{2\pi}\int_h^\infty e^{-\frac{1}{2}x^2}\int_{\frac{k-rx}{\sqrt{(1-r^2)}}}^\infty e^{-\frac{1}{2}y^2}\,dy\,dx,$$

and from this double integral the tables were calculated by quadrature. The table was later (Lee, 1927; Elderton *et al.* 1930) extended to all values of r positive and negative at intervals of 0·05. This is a very extensive table running to more than 20,000 entries; moreover, it is a table of three arguments demanding a not very satisfactory triple interpolation. It is suggested that the table of V given at the end of this paper, of two arguments and extending to no more than 900 entries, would give at least equal accuracy, and for many purposes would be as convenient to use.

3. Calculation of the table

For such a table of V the arguments must obviously be chosen from the sides and angles of the triangle OPH; the side OP may be at once ruled out as requiring far too much preliminary calculation, and for the same reason the angle ϕ or any of its functions is not suitable; this leaves us with the two sides

$$OH = h, \quad \text{and} \quad PH = q = \frac{k - rh}{\sqrt{(1-r^2)}}.$$

While these are suitable arguments, the formula (5) is not suited to them; if, however, we integrate the content of the triangle from the origin we have, using linear variables,

$$V(h,q) = \frac{1}{2\pi}\int_0^h e^{-\frac{1}{2}x^2}\int_0^{qx/h} e^{-\frac{1}{2}y^2}\,dy\,dx, \tag{12}$$

or, using an angular variable,

$$V(h,q) = \frac{1}{2\pi}\int_0^{(\frac{1}{2}\pi-\phi)} \{1 - e^{-\frac{1}{2}(h\sec\psi)^2}\}\,d\psi, \tag{13}$$

and these may be expanded into a form

$$V(h,q) = \frac{1}{2\pi}\left\{B_0\frac{q}{h} - B_1\frac{q^3}{3h^3} + B_2\frac{q^5}{5h^5} - B_3\frac{q^7}{7h^7} + \ldots\right\}, \tag{14}$$

where
$$B_0 = \int_0^h h e^{-\frac{1}{2}h^2} dh, \tag{15}$$
$$B_1 = \frac{1}{2}\int_0^h h^3 e^{-\frac{1}{2}h^2} dh, \tag{15a}$$
.....
$$B_n = \frac{1}{2^n . n!}\int_0^h h^{(2n+1)} e^{-\frac{1}{2}h^2} dh. \tag{15b}$$

The first requisite for calculating V is, then, a table of B_n and
$$B_n = 1 - e^{-\frac{1}{2}h^2}\left\{1 + \frac{h^2}{2} + \frac{h^4}{2^2 . 2!} + \frac{h^6}{2^3 . 3!} + \ldots + \frac{h^{2n}}{2^n . n!}\right\}. \tag{15c}$$

The value actually tabled was $B_n/2\pi(2n+1)$, and since (when $q = h$)
$$\frac{1}{2\pi}\{B_0 - \tfrac{1}{3}B_1 + \tfrac{1}{5}B_2 - \tfrac{1}{7}B_3 + \ldots\} = \frac{1}{2}\left\{\frac{1}{\sqrt{(2\pi)}}\int_0^h e^{-\frac{1}{2}h^2} dh\right\}^2, \tag{16}$$
there is a very useful check on accuracy. This table was taken to 8 places of decimals and the figures for $h = 3$ are given to show the number of terms which are necessary for the higher values of h.

Table 1

n	$B_n/2\pi(2n+1)$	n	$B_n/2\pi(2n+1)$
0	0·15738689	1	0·04981022
2	2630583	3	1495384
4	827423	5	429819
6	206840	7	91871
8	37689	9	14318
10	5054	11	1664
12	513	13	149
14	41	15	11
16	3	17	1
Sum	0·19446835	Sum	0·07014239
	0·07014239		
	$V(3, 3) = 0·12432596$		
	$2V(3, 3) = 0·24865192$		
	$\sqrt{\{2V(3, 3)\}} = 0·49865010$		

$B_n/2\pi(2n+1)$ was then divided by $h^{(2n+1)}$ and the result multiplied by the successive values of $q^{(2n+1)}$; for this purpose it was not necessary to use all the figures of the powers except in the cases of B_0, B_1, and B_2. Beyond that a diminishing number of figures is enough to maintain accuracy in the 8th decimal place.

It is obvious that this method can only be used when $q < h$, so that a diagonal half of the table of V was calculated in this way; for the other half the identity
$$V(h, q) + V(q, h) = \frac{1}{\sqrt{(2\pi)}}\int_0^h e^{-\frac{1}{2}h^2} dh \times \frac{1}{\sqrt{(2\pi)}}\int_0^q e^{-\frac{1}{2}q^2} dq \tag{17}$$
was used. The 8th place of decimals was now discarded and the remaining 7-figure table checked for accuracy in both directions by fourth or even in some cases fifth differences. When it was proved that the error in the 7th place was not greater than 1, the 7th place was discarded and the table completed.

The slope of the table and the magnitude of differences may be gathered from the two differential equations

$$\frac{\partial V}{\partial q} = \frac{1}{2\pi}\frac{h}{h^2+q^2}\{1-e^{-\frac{1}{2}(h^2+q^2)}\}, \tag{18}$$

and

$$\frac{\partial V}{\partial h} = \frac{1}{2\pi}\left\{e^{-\frac{1}{2}h^2}\int_0^q e^{-\frac{1}{2}q^2}dq - \frac{q}{q^2+h^2}(1-e^{-\frac{1}{2}(q^2+h^2)})\right\}. \tag{19}$$

In the first case, first differences while large do not change much within the range of the table, so that second differences are not appreciable. In the second case, first differences are greater at first and moreover change sign within the range of the table, so that second differences are not negligible except for high values of h.

All use of the table demands the measure of an angle (expressed as a fraction of π), and its trigonometric functions; the auxiliary table (Table 7) was therefore added, and, bearing in mind that the main use of the table must be to obtain a value for the κ of equation (11) from r, the argument chosen was r. From r the succeeding columns of the table were calculated in the order given, the angle being calculated as the inverse tangent; the angle was checked by differences and, as an additional check, was calculated by inverse interpolation from a 7-figure table of the sine. Interpolation is satisfactory in all columns of the table as far as $r = 0{\cdot}80$; intermediate values are not often required above this and can usually be obtained by interpolating for $\frac{1}{2}\pi - \lambda$.

4. Notes on the table

The relationship of $V(h, q)$ to d/N is given by the equation

$$V(h, q_h) + V(k, q_k) + \kappa/2\pi = d/N + \frac{1}{2}\frac{1}{\sqrt{(2\pi)}}\int_0^h e^{-\frac{1}{2}h^2}dh + \frac{1}{2}\frac{1}{\sqrt{(2\pi)}}\int_0^k e^{-\frac{1}{2}k^2}dk. \tag{20}$$

Although $V(h, q)$ has been described as the content above the triangle OPH this is not strictly true; when the point P in Fig. 3 lies between the lines $X_hOX'_h$ and $X_kOX'_k$, rh is greater than k so that q, and so also V, is negative, so that strictly speaking V is not a measure of volume but a mathematical conception. As with d/N in the quadrants where h and k are of opposite sign, r is taken to be negative with the deviates both positive. When r is negative κ is replaced by its complement $\pi - \kappa$.

The limiting values of V, when h and q are beyond the range of the table, are necessary for the fitting of a surface.

(1) When $h = 0$, then $B_n = 0$ for all n, and

$$V(0, q) = 0.$$

(2) When h is finite and

(a) $k = rh$, then $q = 0$, and

$$V(h, 0) = 0,$$

(b) $k = \infty$, then $q = \infty$, and

$$V(h, \infty) = \frac{1}{2}\frac{1}{\sqrt{(2\pi)}}\int_0^h e^{-\frac{1}{2}h^2}dh. \tag{21}$$

(3) When $h = \infty$ and

(a) k is finite, then $B_n = 1$, and $\frac{q}{h} = \frac{-r}{\sqrt{(1-r^2)}}$ so that

$$V(\infty, q) = \frac{1}{2\pi}\left\{\tan^{-1}\left(\frac{-r}{\sqrt{(1-r^2)}}\right)\right\} = \frac{1}{2\pi}(\tfrac{1}{2}\pi - \kappa), \tag{22}$$

(b) $k = \infty$, it is not possible to assign a value to $V(h, q_h)$ or $V(k, q_k)$ separately, but their sum

$$V(h, q_h) + V(k, q_k) = \frac{1}{2\pi}(\pi - \kappa). \tag{23}$$

It must sometimes happen, especially when the value of r is high, that q lies outside the range of the table; in such cases we may make use of the identity

$$V(h,q) = \frac{1}{2}\frac{1}{\sqrt{(2\pi)}}\int_0^h e^{-\frac{1}{2}h^2}dh - \frac{1}{2\pi}\tan^{-1}(h/q) + R, \tag{24}$$

where R, geometrically is the content of the sector $H'PP'$ in Fig. 3, and when q is greater than 3 the value of R is negligible. When $q = 3$, the value of R does not exceed 0·0000288, when $q = 4$ R does not exceed 0·0000005.

5. USES OF THE TABLE

There are three main uses of the table.

A. It may be used to calculate the probability integral for the distribution of the ratio between normal variables, v. From the given distribution values are calculated for p as in (1), for the ratio between the principal axes, a/b, and for $\tan\delta = t$. Then taking $\overline{Y}/\overline{X}$ as v_0,

$$\tan\alpha = \frac{v_0 - t}{1 + v_0 t} \tag{25}$$

and

$$\tan(\alpha + \theta_n) = \frac{v_n - t}{1 + v_n t}, \tag{25a}$$

so that it is easy to get out a series of values of ϕ_n by equation (3), corresponding to a series of values of v_n, with the aid of Table 7. The table of V is then entered with $h = p\sin\phi$, and $q = p\cos\phi$, and

$$P\{v_0 \leqslant v \leqslant v_n\} = \phi_n/\pi + 2V(p\sin\phi_n,\ p\cos\phi_n). \tag{26}$$

B. For most uses of the table it is of course necessary to evaluate both $V(h, q_h)$, and $V(k, q_k)$; it is the sum of these which we may designate $W(h, k, r)$ which is to be used. In obtaining a correlation coefficient from a fourfold table of supposedly normal distribution we must use the equation

$$(a+d)/N = \kappa/\pi + 2W(h,k,r). \tag{27}$$

Inverse interpolation is tedious, but

$$\frac{d\{(a+d)/N\}}{dr} = \frac{1}{\pi\sqrt{(1-r^2)}}\,e^{-\frac{1}{2}p^2} \tag{28}$$

which may readily be calculated as

$$\frac{2}{\sqrt{(1-r^2)}} \times \frac{1}{\sqrt{(2\pi)}}\,e^{-\frac{1}{2}h^2} \times \frac{1}{\sqrt{(2\pi)}}\,e^{-\frac{1}{2}q^2}, \tag{28a}$$

so that having obtained an approximate solution less than the true solution it is possible to obtain further accuracy by the use of the Newton-Raphson method of solving equations. It should be noted that most of the value of $(a+d)/N$, when h and k are not large, is carried by κ/π.

C. The main use of the table is no doubt the fitting of a normal surface to a given distribution, and the lay-out for this purpose is illustrated in Fig. 4. It will be seen that $W(h, k, r)$, being the content of the quadrilateral $OHPK$, may be used in much the same way as d/N for obtaining the content of any cell; thus the cell $P_{12}P_{13}P_{17}P_{18}$ is given by $W_{13} + W_{17} - W_{12} - W_{18}$, the cell $P_{13}P_{14}P_{18}P_{19}$ by $W_{18} + W_{19} - W_{13} - W_{14}$, the cell containing the origin is given by the sum of the W's for the surrounding four nodes. It will be seen that in the case of a 4×4-fold table it is necessary to get values of $W(h, k, r)$ for 25 nodes, but of these the table of $V(h, q)$ is used for 9 only, the remainder are functions of $\kappa/2\pi$ and of the

probability integrals for h and k, so that 18 double interpolations are required. In the case of the table of d/N, again 25 values of d/N are to be calculated, but again 9 only of these demand triple interpolation, most of the remainder demand double interpolation. On the whole, then, the table of $V(h, q)$ has no great advantage over the table of d/N except that it is so much more compact.

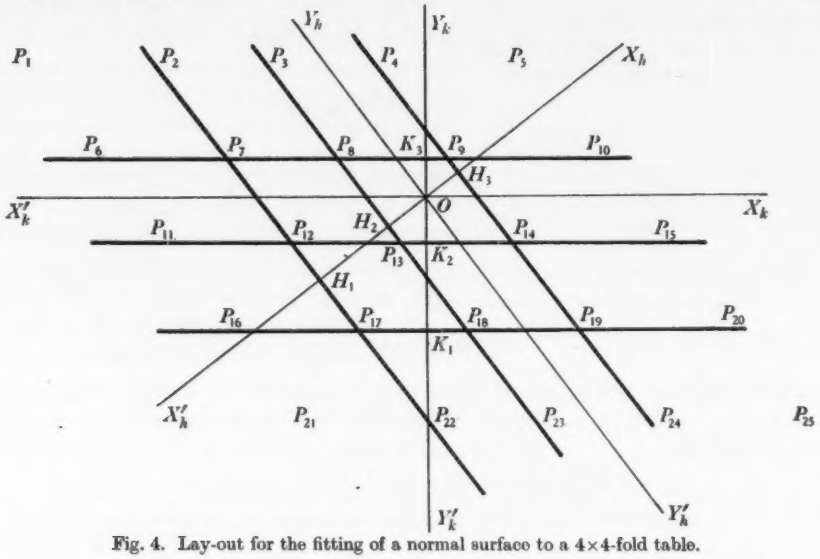

Fig. 4. Lay-out for the fitting of a normal surface to a 4×4-fold table.

6. Example*

The classification of the female pelvis according to the shape of the brim (illustrated in Fig. 5) has hitherto depended on the pelvic index, the percentage ratio of the antero-posterior to the greatest transverse diameter, $100AB : CD$, following Turner (1886). Caldwell & Moloy (1938) have reintroduced a second criterion of classification depending on the relative position on the antero-posterior diameter of the point where it is crossed by the transverse diameter; this may be described by the sagittal index, the percentage ratio of the posterior part to the whole antero-posterior diameter, $100AO : AB$. Measurements in 329 cases made by an accurate technique of X-ray pelvimetry (Nicholson, 1936) give the following constants for these two indices:

	Mean	Standard deviation
Pelvic index	89·92	8·63
Sagittal index	34·69	4·44

As with other anatomical measurements, the distribution of these indices fits well to the normal curve; this may be appreciated from the marginal frequencies in Table 2. The correlation coefficient between them is 0·40. Now it is claimed that a low value of the

* The figures used in this example are taken from an inquiry into the value of X-ray pelvimetry in obstetrics which is assisted by a grant from the Medical Research Council.

sagittal index is due to an imbalance of the sex hormones with male predominance, and Caldwell & Moloy rather beg the question by naming the pelvis of this type the 'android' pelvis. The normality of the distribution of the sagittal index is already a strong argument against this theory, which would be reinforced if it were proved that the correlation between the indices was normal correlation. Further, it is necessary to establish this normality before discussing the frequency of any of Caldwell & Moloy's types. The 329 cases are arranged below in a 4 × 4-fold table (Table 2), the figures in brackets are the theoretical normal frequencies and the procedure used in calculating these is shown in the succeeding Tables, 3–5.

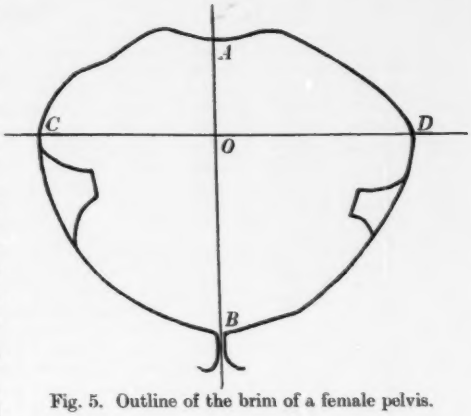

Fig. 5. Outline of the brim of a female pelvis.

Table 2

		Pelvic index				
		76·5	88·5	100·5		
Sagittal index	40·5	0 (0·0)	3 (5·3)	20 (16·8)	6 (8·9)	29 (31·0)
	34·5	5 (4·1)	44 (43·3)	76 (72·2)	22 (19·5)	147 (139·1)
	28·5	8 (10·5)	54 (60·1)	55 (53·9)	9 (7·4)	126 (131·9)
		8 (5·1)	15 (14·6)	4 (6·8)	0 (0·4)	27 (26·9)
		21 (19·7)	116 (123·3)	155 (149·7)	37 (36·2)	329 (328·9)

$r = 0{\cdot}40, \quad \kappa/\pi = 0{\cdot}6310$

Table 3

	h	$\frac{h}{\sqrt{(1-r^2)}}$	$\frac{rh}{\sqrt{(1-r^2)}}$	$\frac{1}{2}\frac{1}{\sqrt{(2\pi)}}\int_0^h e^{-\frac{1}{2}h^2}dh$
h_1	−1·556	1·698	0·679	0·2201
h_2	−0·165	0·180	0·072	0·0327
h_3	1·226	1·338	0·535	0·1949
k_1	−1·394	1·521	0·608	0·2092
k_2	−0·043	0·047	0·019	0·0086
k_3	1·308	1·427	0·571	0·2028

Table 4

P	h	q_h	$V(h, q_h)$	k	q_k	$V(k, q_k)$	$W(h, k, r)$
1	$-\infty$	—	—	∞	—	—	0·3155
2	−1·556	∞	0·2201	∞	—	0·0655	0·2856
3	−0·165	∞	0·0327	∞	—	0·0655	0·0982
4	1·226	∞	0·1949	∞	—	−0·0655	0·1294
5	∞	—	—	∞	—	—	0·1845
6	$-\infty$	—	0·0655	1·308	∞	0·2028	0·2683
7	−1·556	2·106	0·1191	1·308	2·269	0·1193	0·2384
8	−0·165	1·499	0·0166	1·308	0·751	0·0505	0·0671
9	1·226	0·892	0·0578	1·308	0·767	0·0515	0·1093
10	∞	—	−0·0655	1·308	∞	0·2028	0·1373
11	$-\infty$	—	−0·0655	−0·043	∞	0·0086	−0·0569
12	−1·556	−0·632	−0·0442	−0·043	1·679	0·0047	−0·0395
13	−0·165	−0·025	−0·0003	−0·043	0·161	0·0005	0·0002
14	1·226	0·582	0·0389	−0·043	1·357	0·0040	0·0429
15	∞	—	0·0655	−0·043	∞	0·0086	0·0741
16	$-\infty$	—	−0·0655	−1·394	∞	0·2092	0·1437
17	−1·556	0·842	0·0577	−1·394	1·090	0·0715	0·1292
18	−0·165	1·449	0·0161	−1·394	−0·428	−0·0300	−0·0139
19	1·226	2·056	0·1099	−1·394	1·946	0·1110	0·2209
20	∞	—	0·0655	−1·394	∞	0·2092	0·2747
21	$-\infty$	—	—	$-\infty$	—	—	0·1845
22	−1·556	∞	0·2201	$-\infty$	—	−0·0655	0·1546
23	−0·165	∞	0·0327	$-\infty$	—	−0·0655	−0·0328
24	1·226	∞	0·1949	$-\infty$	—	0·0655	0·2604
25	∞	—	—	$-\infty$	—	—	0·3155

Table 5

Cell	Expected	Observed	χ^2
$a = W(\ 1+\ 7-\ 2-\ 6) = 0·0000$	0·0	0	1·00
$b = W(\ 2+\ 8-\ 3-\ 7) = 0·0161$	5·3	3	
$c = W(\ 3+\ 4-\ 8-\ 9) = 0·0512$	16·8	20	0·61
$d = W(\ 5+\ 9-\ 4-10) = 0·0271$	8·9	6	0·94
$e = W(\ 6+11-\ 7-12) = 0·0125$	4·1	5	0·20
$f = W(\ 7+12-\ 8-13) = 0·1316$	43·3	44	0·01
$g = W(\ 8+13+\ 9+14) = 0·2195$	72·2	76	0·20
$h = W(10+15-\ 9-14) = 0·0592$	19·5	22	0·32
$i = W(12+16-11-17) = 0·0319$	10·5	8	0·60
$j = W(13+17-12-18) = 0·1828$	60·1	54	0·62
$k = W(18+19-13-14) = 0·1639$	53·9	55	0·02
$l = W(14+20-15-19) = 0·0226$	7·4	9	0·35
$m = W(17+21-16-22) = 0·0154$	5·1	8	1·65
$n = W(18+22-17-23) = 0·0443$	14·6	15	0·01
$o = W(23+24-18-19) = 0·0206$	6·8	4	1·42
$p = W(19+25-20-24) = 0·0013$	0·4	0	
1·0000	328·9	329	7·95

Taking the degrees of freedom as 8, this gives a probability of nearly 0·40 of obtaining a worse fit through chance fluctuations, so that we may take it that normal correlation is a good description of this distribution.

Table 6. *Table of $V(h, q)$*

h \ q	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0	q / h
0·1	·000793	·001582	·002364	·003134	·003888	·004625	·005340	·006032	·006699	·007338	0·1
0·2	·001574	·003141	·004692	·006221	·007719	·009182	·010602	·011976	·013300	·014569	0·2
0·3	·002333	·004653	·006952	·009216	·011437	·013604	·015710	·017746	·019708	·021590	0·3
0·4	·003057	·006098	·009110	·012078	·014989	·017830	·020591	·023262	·025836	·028305	0·4
0·5	·003737	·007456	·011139	·014769	·018329	·021805	·025184	·028453	·031604	·034628	0·5
0·6	·004366	·008711	·013014	·017256	·021417	·025481	·029432	·033256	·036943	·040483	0·6
0·7	·004937	·009850	·014716	·019513	·024221	·028819	·033291	·037622	·041799	·045811	0·7
0·8	·005444	·010862	·016229	·021522	·026716	·031792	·036730	·041514	·046130	·050567	0·8
0·9	·005885	·011742	·017545	·023268	·028887	·034380	·039725	·044907	·049909	·054720	0·9
1·0	·006258	·012486	·018659	·024747	·030727	·036574	·042268	·047790	·053124	·058258	1·0
1·1	·006563	·013096	·019571	·025960	·032237	·038378	·044360	·050165	·055777	·061182	1·1
1·2	·006802	·013575	·020288	·026914	·033425	·039799	·046011	·052044	·057880	·063505	1·2
1·3	·006979	·013928	·020817	·027619	·034307	·040855	·047243	·053449	·059458	·065255	1·3
1·4	·007097	·014164	·021172	·028093	·034901	·041571	·048081	·054411	·060544	·066468	1·4
1·5	·007161	·014293	·021366	·028355	·035232	·041974	·048559	·054966	·061180	·067187	1·5
1·6	·007177	·014325	·021418	·028427	·035328	·042097	·048712	·055155	·061409	·067461	1·6
1·7	·007150	·014274	·021342	·028331	·035215	·041972	·048580	·055022	·061281	·067343	1·7
1·8	·007088	·014149	·021159	·028092	·034925	·041635	·048203	·054611	·060843	·066887	1·8
1·9	·006995	·013965	·020885	·027732	·034484	·041120	·047619	·053966	·060145	·066144	1·9
2·0	·006877	·013730	·020537	·027274	·033921	·040458	·046866	·053130	·059234	·065167	2·0
2·1	·006740	·013457	·020130	·026739	·033262	·039682	·045980	·052141	·058153	·064002	2·1
2·2	·006588	·013154	·019680	·026145	·032530	·038818	·044992	·051037	·056941	·062693	2·2
2·3	·006425	·012831	·019199	·025509	·031746	·037891	·043930	·049849	·055636	·061279	2·3
2·4	·006256	·012494	·018697	·024847	·030928	·036924	·042820	·048605	·054267	·059795	2·4
2·5	·006084	·012151	·018185	·024170	·030091	·035933	·041684	·047330	·052862	·058269	2·5
2·6	·005910	·011805	·017670	·023489	·029249	·034936	·040538	·046044	·051442	·056725	2·6
2·7	·005738	·011462	·017158	·022813	·028412	·033944	·039398	·044762	·050027	·055185	2·7
2·8	·005569	·011125	·016655	·022147	·027588	·032968	·038274	·043498	·048630	·053662	2·8
2·9	·005404	·010796	·016165	·021498	·026784	·032013	·037175	·042261	·047262	·052171	2·9
3·0	·005244	·010478	·015689	·020868	·026004	·031087	·036109	·041059	·045932	·050719	3·0

h \ q	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	q / h
0·1	·007948	·008529	·009080	·009601	·010092	·010555	·010989	·011395	·011776	·012131	0·1
0·2	·015781	·016935	·018030	·019066	·020042	·020961	·021824	·022633	·023389	·024097	0·2
0·3	·023388	·025100	·026725	·028262	·029712	·031077	·032359	·033560	·034685	·035736	0·3
0·4	·030665	·032913	·035047	·037066	·038973	·040767	·042453	·044035	·045516	·046901	0·4
0·5	·037519	·040274	·042891	·045368	·047708	·049912	·051983	·053927	·055749	·057454	0·5
0·6	·043870	·047098	·050166	·053072	·055818	·058406	·060841	·063127	·065271	·067280	0·6
0·7	·049652	·053315	·056797	·060099	·063221	·066166	·068938	·071544	·073989	·076281	0·7
0·8	·054816	·058872	·062731	·066392	·069856	·073127	·076209	·079108	·081832	·084387	0·8
0·9	·059331	·063735	·067929	·071911	·075683	·079248	·082609	·085775	·088752	·091548	0·9
1·0	·063182	·067889	·072375	·076639	·080682	·084506	·088117	·091521	·094726	·097740	1·0
1·1	·066370	·071334	·076069	·080575	·084852	·088902	·092731	·096345	·099752	·102960	1·1
1·2	·068910	·074086	·079029	·083737	·088212	·092454	·096470	·100266	·103849	·107227	1·2
1·3	·070830	·076175	·081285	·086158	·090795	·095197	·099370	·103320	·107053	·110579	1·3
1·4	·072169	·077642	·082881	·087883	·092648	·097179	·101480	·105557	·109416	·113066	1·4
1·5	·072975	·078537	·083868	·088965	·093828	·098458	·102860	·107039	·111001	·114754	1·5
1·6	·073300	·078917	·084307	·089469	·094400	·099102	·103579	·107836	·111879	·115716	1·6
1·7	·073199	·078840	·084261	·089458	·094431	·099181	·103710	·108024	·112128	·116029	1·7
1·8	·072731	·078369	·083793	·089002	·093993	·098768	·103329	·107680	·111827	·115775	1·8
1·9	·071953	·077563	·082968	·088167	·093156	·097937	·102511	·106882	·111054	·115034	1·9
2·0	·070918	·076481	·081848	·087018	·091987	·096756	·101327	·105703	·109886	·113884	2·0
2·1	·069679	·075178	·080491	·085616	·090550	·095294	·099847	·104214	·108396	·112399	2·1
2·2	·068283	·073704	·078950	·084017	·088904	·093609	·098134	·102479	·106649	·110647	2·2
2·3	·066771	·072104	·077272	·082271	·087100	·091757	·096243	·100558	·104706	·108689	2·3
2·4	·065181	·070417	·075499	·080423	·085186	·089786	·094225	·098502	·102620	·106581	2·4
2·5	·063543	·068678	·073668	·078510	·083201	·087738	·092123	·096355	·100436	·104368	2·5
2·6	·061885	·066914	·071809	·076564	·081178	·085648	·089973	·094155	·098194	·102092	2·6
2·7	·060228	·065150	·069945	·074612	·079145	·083544	·087806	·091934	·095927	·099785	2·7
2·8	·058589	·063402	·068099	·072674	·077125	·081449	·085647	·089717	·093660	·097476	2·8
2·9	·056981	·061687	·066283	·070767	·075134	·079384	·083514	·087524	·091414	·095185	2·9
3·0	·055415	·060014	·064511	·068903	·073187	·077360	·081421	·085370	·089206	·092929	3·0

Table 6 (*continued*)

h \ q	2·1	2·2	2·3	2·4	2·5	2·6	2·7	2·8	2·9	3·0	∞	h \ q
0·1	·012463	·012773	·013062	·013331	·013583	·013818	·014038	·014243	·014435	·014616	·019914	0·1
0·2	·024757	·025374	·025949	·026486	·026987	·027455	·027893	·028302	·028686	·029045	·039630	0·2
0·3	·036719	·037636	·038493	·039292	·040039	·040736	·041389	·041999	·042571	·043107	·058956	0·3
0·4	·048195	·049405	·050535	·051590	·052576	·053497	·054359	·055167	·055923	·056633	·077711	0·4
0·5	·059049	·060539	·061932	·063234	·064451	·065590	·066655	·067654	·068590	·069469	·095731	0·5
0·6	·069159	·070917	·072561	·074099	·075538	·076885	·078147	·079329	·080439	·081482	·112873	0·6
0·7	·078429	·080439	·082321	·084083	·085732	·087277	·088726	·090085	·091361	·092561	·129018	0·7
0·8	·086783	·089029	·091133	·093105	·094953	·096686	·098312	·099839	·101274	·102624	·144072	0·8
0·9	·094173	·096636	·098946	·101113	·103146	·105055	·106848	·108533	·110118	·111612	·157970	0·9
1·0	·100573	·103234	·105733	·108080	·110284	·112356	·114304	·116138	·117865	·119492	·170672	1·0
1·1	·105979	·108819	·111489	·114000	·116362	·118584	·120676	·122648	·124506	·126260	·182167	1·1
1·2	·110411	·113410	·116233	·118893	·121397	·123757	·125981	·128079	·130060	·131932	·192465	1·2
1·3	·113906	·117044	·120004	·122795	·125428	·127912	·130257	·132471	·134564	·136545	·201600	1·3
1·4	·116516	·119776	·122854	·125762	·128508	·131104	·133557	·135877	·138073	·140153	·209622	1·4
1·5	·118308	·121670	·124851	·127860	·130706	·133400	·135950	·138365	·140654	·142825	·216596	1·5
1·6	·119354	·122802	·126069	·129164	·132098	·134878	·137513	·140013	·142386	·144639	·222600	1·6
1·7	·119734	·123252	·126590	·129759	·132766	·135621	·138332	·140907	·143354	·145681	·227717	1·7
1·8	·119531	·123104	·126500	·129729	·132798	·135717	·138492	·141132	·143645	·146038	·232035	1·8
1·9	·118827	·122440	·125882	·129159	·132279	·135251	·138081	·140778	·143348	·145800	·235642	1·9
2·0	·117700	·121343	·124818	·128132	·131293	·134309	·137185	·139930	·142550	·145051	·238625	2·0
2·1	·116227	·119888	·123385	·126727	·129919	·132969	·135884	·138668	·141331	·143876	·241068	2·1
2·2	·114477	·118145	·121656	·125016	·128232	·131308	·134252	·137070	·139767	·142350	·243048	2·2
2·3	·112512	·116179	·119696	·123066	·126296	·129392	·132359	·135203	·137929	·140543	·244638	2·3
2·4	·110389	·114047	·117561	·120935	·124173	·127282	·130265	·133129	·135878	·138517	·245901	2·4
2·5	·108154	·111798	·115303	·118674	·121915	·125029	·128024	·130902	·133668	·136328	·246895	2·5
2·6	·105851	·109474	·112965	·116327	·119564	·122680	·125680	·128567	·131346	·134021	·247669	2·6
2·7	·103513	·107111	·110582	·113931	·117160	·120272	·123273	·126164	·128951	·131637	·248267	2·7
2·8	·101168	·104736	·108185	·111516	·114732	·117837	·120834	·123726	·126517	·129210	·248722	2·8
2·9	·098838	·102374	·105796	·109106	·112306	·115399	·118389	·121278	·124069	·126766	·249067	2·9
3·0	·096541	·100042	·103435	·106720	·109901	·112980	·115959	·118841	·121629	·124326	·249325	3·0

$$V(0, q)=0, \quad V(h, 0)=0, \quad V(\infty, q)=\frac{1}{2\pi}\tan^{-1}\left\{\frac{-r}{\sqrt{(1-r^2)}}\right\}, \quad V(h, \infty)=\frac{1}{2}\frac{1}{\sqrt{(2\pi)}}\int_0^h e^{-\frac{1}{2}h^2}\,dh,$$

$$W(\infty, \infty, r)=\frac{1}{\pi}\tan^{-1}\left\{\frac{\sqrt{(1-r)}}{\sqrt{(1+r)}}\right\}, \quad \text{for } q>3\ V(h, q)=\frac{1}{2}\frac{1}{\sqrt{(2\pi)}}\int_0^h e^{-\frac{1}{2}h^2}.dh-\frac{1}{2\pi}\tan^{-1}(h/q).$$

Table 7. *Auxiliary table of trigonometric functions*

r	$\sqrt{(1-r^2)}$	$1/\sqrt{(1-r^2)}$	$r/\sqrt{(1-r^2)}$	$\kappa/\pi-\frac{1}{2}$	r	$\sqrt{(1-r^2)}$	$1/\sqrt{(1-r^2)}$	$r/\sqrt{(1-r^2)}$	$\kappa/\pi-\frac{1}{2}$
$\sin\lambda$	$\cos\lambda$	$\sec\lambda$	$\tan\lambda$	λ/π	$\sin\lambda$	$\cos\lambda$	$\sec\lambda$	$\tan\lambda$	λ/π
0·00	1·00000	1·00000	0·00000	0·000000	**0·50**	0·86603	1·15470	0·57735	0·166667
0·01	0·99995	1·00005	0·01000	0·003183	**0·51**	0·86017	1·16255	0·59290	0·170355
·02	·99980	1·00020	·02000	·006367	·52	·85417	1·17073	·60878	·174068
·03	·99955	1·00045	·03001	·009551	·53	·84800	1·17925	·62500	·177808
·04	·99920	1·00080	·04003	·012736	·54	·84167	1·18812	·64159	·181576
·05	·99875	1·00125	·05006	·015922	·55	·83516	1·19737	·65855	·185373
0·06	0·99820	1·00180	0·06011	0·019110	**0·56**	0·82849	1·20701	0·67593	0·189199
·07	·99755	1·00246	·07017	·022300	·57	·82164	1·21707	·69373	·193057
·08	·99679	1·00322	·08026	·025492	·58	·81462	1·22757	·71199	·196948
·09	·99599	1·00407	·09037	·028687	·59	·80740	1·23854	·73074	·200872
·10	·99499	1·00504	·10050	·031884	·60	·80000	1·25000	·75000	·204833
0·11	0·99393	1·00611	0·11067	0·035085	**0·61**	0·79240	1·26199	0·76981	0·208831
·12	·99277	1·00728	·12087	·038289	·62	·78460	1·27453	·79021	·212867
·13	·99151	1·00856	·13111	·041498	·63	·77660	1·28767	·81123	·216945
·14	·99015	1·00995	·14139	·044710	·64	·76837	1·30145	·83293	·221066
·15	·98869	1·01144	·15172	·047927	·65	·75993	1·31590	·85534	·225231
0·16	0·98712	1·01305	0·16209	0·051150	**0·66**	0·75127	1·33109	0·87852	0·229443
·17	·98544	1·01477	·17251	·054377	·67	·74236	1·34705	·90253	·233706
·18	·98367	1·01660	·18299	·057610	·68	·73321	1·36386	·92743	·238020
·19	·98178	1·01855	·19353	·060849	·69	·72381	1·38158	·95329	·242390
·20	·97980	1·02062	·20412	·064094	·70	·71414	1·40028	·98020	·246817
0·21	0·97770	1·02281	0·21479	0·067346	**0·71**	0·70420	1·42005	1·00823	0·251305
·22	·97550	1·02512	·22553	·070606	·72	·69397	1·44098	1·03750	·255858
·23	·97319	1·02755	·23634	·073873	·73	·68345	1·46317	1·06811	·260480
·24	·97077	1·03011	·24723	·077147	·74	·67261	1·48675	1·10020	·265174
·25	·96825	1·03280	·25820	·080431	·75	·66144	1·51186	1·13389	·269946
0·26	0·96561	1·03562	0·26926	0·083723	**0·76**	0·64992	1·53864	1·16937	0·274801
·27	·96280	1·03857	·28041	·087024	·77	·63804	1·56729	1·20681	·279744
·28	·96000	1·04167	·29167	·090335	·78	·62578	1·59801	1·24645	·284781
·29	·95703	1·04490	·30302	·093655	·79	·61311	1·63104	1·28852	·289919
·30	·95394	1·04828	·31449	·096987	·80	·60000	1·66667	1·33333	·295167
0·31	0·95074	1·05182	0·32606	0·100329	**0·81**	0·58643	1·70523	1·38124	0·300533
·32	·94742	1·05550	·33776	·103683	·82	·57236	1·74714	1·43266	·306027
·33	·94398	1·05934	·34958	·107049	·83	·55776	1·79287	1·48809	·311660
·34	·94043	1·06335	·36154	·110427	·84	·54259	1·84302	1·54814	·317445
·35	·93675	1·06752	·37363	·113818	·85	·52678	1·89832	1·61357	·323398
0·36	0·93295	1·07187	0·38587	0·117223	**0·86**	0·51029	1·95965	1·68530	0·329537
·37	·92903	1·07639	·39826	·120642	·87	·49305	2·02818	1·76452	·335881
·38	·92499	1·08110	·41082	·124076	·88	·47497	2·10358	1·85273	·342458
·39	·92081	1·08599	·42354	·127525	·89	·45596	2·19317	1·95192	·349296
·40	·91652	1·09109	·43644	·130990	·90	·43589	2·29416	2·06474	·356433
0·41	0·91209	1·09639	0·44952	0·134471	**0·91**	0·41461	2·41192	2·19484	0·363919
·42	·90752	1·10190	·46280	·137970	·92	·39192	2·55155	2·34743	·371812
·43	·90283	1·10763	·47628	·141487	·93	·36756	2·72065	2·53020	·380193
·44	·89800	1·11359	·48998	·145022	·94	·34117	2·93105	2·75519	·389175
·45	·89303	1·11979	·50390	·148576	·95	·31225	3·20256	3·04243	·398917
0·46	0·88792	1·12623	0·51807	0·152151	**0·96**	0·28000	3·57143	3·42857	0·409666
·47	·88267	1·13293	·53248	·155746	·97	·24310	4·11343	3·99003	·421834
·48	·87727	1·13990	·54715	·159363	·98	·19900	5·02519	4·92469	·436231
·49	·87172	1·14715	·56211	·163003	·99	·14107	7·08881	7·01792	·454947
·50	·86603	1·15470	·57735	·166667	**1·00**	·00000	∞	∞	·500000

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TABLES OF PERCENTAGE POINTS OF THE INVERTED BETA (F) DISTRIBUTION

COMPUTED BY MAXINE MERRINGTON AND CATHERINE M. THOMPSON

PREFATORY NOTE BY E. S. PEARSON

The following tables of the percentage points of F, using the notation of Snedecor (1934), express the results of Miss Thompson's tabulation (1941*a*) of the incomplete beta function in terms of the argument most convenient for use in the analysis of variance.

If we take the elementary probability function* of the beta distribution, namely

$$f(x) = \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)}\, x^{p-1}(1-x)^{q-1} \tag{1}$$

and make the transformation

$$x = \frac{1}{1+u}, \tag{2}$$

we obtain for u the inverted beta distribution

$$f(u) = \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)}\, u^{q-1}(1+u)^{-p-q}. \tag{3}$$

The limits 0 and 1 for x correspond to limits of ∞ and 0 for u. While equation (1) represents a standardized form of Karl Pearson's Type I frequency distribution, (3) is a form of his Type VI. The *Tables of Incomplete Beta-Function* (K. Pearson 1934) provide the probability integral of (1) and therefore of (3).

In the terminology of the analysis of variance let S_1 and S_2 be two sums of squares of normal variates having, respectively, ν_1 and ν_2 degrees of freedom. If all the variates have a common standard deviation σ and if S_1 and S_2 are independent, then it is known that:

(*a*) S_1/σ^2 and S_2/σ^2 are distributed in the standard χ^2 form, namely

$$f(\chi^2) = \frac{(\frac{1}{2})^{\frac{1}{2}\nu}}{\Gamma(\frac{1}{2}\nu)}\, (\chi^2)^{\frac{1}{2}\nu-1}\, e^{-\frac{1}{2}\chi^2} \tag{4}$$

with ν_1 and ν_2 degrees of freedom, respectively.

(*b*) The ratio S_1/S_2 is distributed as u in (3), where $q = \frac{1}{2}\nu_1$, $p = \frac{1}{2}\nu_2$.

(*c*) Writing

$$s_1^2 = \frac{S_1}{\nu_1}, \quad s_2^2 = \frac{S_2}{\nu_2} \tag{5}$$

as two independent estimates of σ^2, the ratio

$$F = \frac{s_1^2}{s_2^2} = \frac{\nu_2 S_1}{\nu_1 S_2} \tag{6}$$

has a probability distribution

$$f(F) = \frac{\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)}{\Gamma(\frac{1}{2}\nu_1)\,\Gamma(\frac{1}{2}\nu_2)}\, \nu_1^{\frac{1}{2}\nu_1}\nu_2^{\frac{1}{2}\nu_2} F^{\frac{1}{2}\nu_1-1}(\nu_2+\nu_1 F)^{-\frac{1}{2}(\nu_1+\nu_2)}. \tag{7}$$

It is useful to note that, for (7),

$$\text{Expectation of } F = \frac{\nu_2}{\nu_2-2}, \qquad \text{for } \nu_2 > 2, \tag{8}$$

$$\sigma_F = \frac{\nu_2}{\nu_2-2}\sqrt{\left\{\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}\right\}}, \qquad \text{for } \nu_2 > 4. \tag{9}$$

Thus for large values of ν_2, F tends to be distributed as χ^2/ν_1 with a mean of unity and standard deviation of $\sqrt{(2/\nu_1)}$.

* The letter f will be used as a general symbol for an elementary probability function, in place of p which might here be confused with the index. The integral probability function for the beta distribution is then

$$P\{0 \leqslant x \leqslant X\} = \int_0^X f(x)\,dx.$$

F Distribution: 50 per cent Points

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9
1	1·0000	1·5000	1·7092	1·8227	1·8937	1·9422	1·9774	2·0041	2·0250
2	0·66667	1·0000	1·1349	1·2071	1·2519	1·2824	1·3045	1·3213	1·3344
3	·58506	0·88110	1·0000	1·0632	1·1024	1·1289	1·1482	1·1627	1·1741
4	·54863	·82843	0·94054	1·0000	1·0367	1·0617	1·0797	1·0933	1·1040
5	0·52807	0·79877	0·90715	0·96456	1·0000	1·0240	1·0414	1·0545	1·0648
6	·51489	·77976	·88578	·94191	0·97654	1·0000	1·0169	1·0298	1·0398
7	·50572	·76655	·87095	·92619	·96026	0·98334	1·0000	1·0126	1·0224
8	·49898	·75683	·86004	·91464	·94831	·97111	0·98757	1·0000	1·0097
9	·49382	·74938	·85168	·90580	·93916	·96175	·97805	0·99037	1·0000
10	0·48973	0·74349	0·84508	0·89882	0·93193	0·95436	0·97054	0·98276	0·99232
11	·48644	·73872	·83973	·89316	·92608	·94837	·96445	·97661	·98610
12	·48369	·73477	·83530	·88848	·92124	·94342	·95943	·97152	·98097
13	·48141	·73145	·83159	·88454	·91718	·93926	·95520	·96724	·97665
14	·47944	·72862	·82842	·88119	·91371	·93573	·95161	·96360	·97298
15	0·47775	0·72619	0·82569	0·87830	0·91073	0·93267	0·94850	0·96046	0·96981
16	·47628	·72406	·82330	·87578	·90812	·93001	·94580	·95773	·96705
17	·47499	·72219	·82121	·87357	·90584	·92767	·94342	·95532	·96462
18	·47385	·72053	·81936	·87161	·90381	·92560	·94132	·95319	·96247
19	·47284	·71906	·81771	·86987	·90200	·92375	·93944	·95129	·96056
20	0·47192	0·71773	0·81621	0·86830	0·90038	0·92210	0·93776	0·94959	0·95884
21	·47108	·71653	·81487	·86688	·89891	·92060	·93624	·94805	·95728
22	·47033	·71545	·81365	·86559	·89759	·91924	·93486	·94665	·95588
23	·46965	·71446	·81255	·86442	·89638	·91800	·93360	·94538	·95459
24	·46902	·71356	·81153	·86335	·89527	·91687	·93245	·94422	·95342
25	0·46844	0·71272	0·81061	0·86236	0·89425	0·91583	0·93140	0·94315	0·95234
26	·46793	·71195	·80975	·86145	·89331	·91487	·93042	·94217	·95135
27	·46744	·71124	·80894	·86061	·89244	·91399	·92952	·94126	·95044
28	·46697	·71059	·80820	·85983	·89164	·91317	·92869	·94041	·94958
29	·46654	·70999	·80753	·85911	·89089	·91241	·92791	·93963	·94879
30	0·46616	0·70941	0·80689	0·85844	0·89019	0·91169	0·92719	0·93889	0·94805
40	·46330	·70531	·80228	·85357	·88516	·90654	·92197	·93361	·94272
60	·46053	·70122	·79770	·84873	·88017	·90144	·91679	·92838	·93743
120	·45774	·69717	·79314	·84392	·87521	·89637	·91164	·92318	·93218
∞	·45494	·69315	·78866	·83918	·87029	·89135	·90654	·91802	·92698

This table gives the values of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}50$.

F DISTRIBUTION: 50 PER CENT POINTS

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	2·0419	2·0674	2·0931	2·1190	2·1321	2·1452	2·1584	2·1716	2·1848	2·1981
2	1·3450	1·3610	1·3771	1·3933	1·4014	1·4096	1·4178	1·4261	1·4344	1·4427
3	1·1833	1·1972	1·2111	1·2252	1·2322	1·2393	1·2464	1·2536	1·2608	1·2680
4	1·1126	1·1255	1·1386	1·1517	1·1583	1·1649	1·1716	1·1782	1·1849	1·1916
5	1·0730	1·0855	1·0980	1·1106	1·1170	1·1234	1·1297	1·1361	1·1426	1·1490
6	1·0478	1·0600	1·0722	1·0845	1·0907	1·0969	1·1031	1·1093	1·1156	1·1219
7	1·0304	1·0423	1·0543	1·0664	1·0724	1·0785	1·0846	1·0908	1·0969	1·1031
8	1·0175	1·0293	1·0412	1·0531	1·0591	1·0651	1·0711	1·0771	1·0832	1·0893
9	1·0077	1·0194	1·0311	1·0429	1·0489	1·0548	1·0608	1·0667	1·0727	1·0788
10	1·0000	1·0116	1·0232	1·0349	1·0408	1·0467	1·0526	1·0585	1·0645	1·0705
11	0·99373	1·0052	1·0168	1·0284	1·0343	1·0401	1·0460	1·0519	1·0578	1·0637
12	·98856	1·0000	1·0115	1·0231	1·0289	1·0347	1·0405	1·0464	1·0523	1·0582
13	·98421	0·99560	1·0071	1·0186	1·0243	1·0301	1·0360	1·0418	1·0476	1·0535
14	·98051	·99186	1·0033	1·0147	1·0205	1·0263	1·0321	1·0379	1·0437	1·0495
15	0·97732	0·98863	1·0000	1·0114	1·0172	1·0229	1·0287	1·0345	1·0403	1·0461
16	·97454	·98582	0·99716	1·0086	1·0143	1·0200	1·0258	1·0315	1·0373	1·0431
17	·97209	·98334	·99466	1·0060	1·0117	1·0174	1·0232	1·0289	1·0347	1·0405
18	·96993	·98116	·99245	1·0038	1·0095	1·0152	1·0209	1·0267	1·0324	1·0382
19	·96800	·97920	·99047	1·0018	1·0075	1·0132	1·0189	1·0246	1·0304	1·0361
20	0·96626	0·97746	0·98870	1·0000	1·0057	1·0114	1·0171	1·0228	1·0285	1·0343
21	·96470	·97587	·98710	0·99838	1·0040	1·0097	1·0154	1·0211	1·0268	1·0326
22	·96328	·97444	·98565	·99692	1·0026	1·0082	1·0139	1·0196	1·0253	1·0311
23	·96199	·97313	·98433	·99558	1·0012	1·0069	1·0126	1·0183	1·0240	1·0297
24	·96081	·97194	·98312	·99436	1·0000	1·0057	1·0113	1·0170	1·0227	1·0284
25	0·95972	0·97084	0·98201	0·99324	0·99887	1·0045	1·0102	1·0159	1·0215	1·0273
26	·95872	·96983	·98099	·99220	·99783	1·0035	1·0091	1·0148	1·0205	1·0262
27	·95779	·96889	·98004	·99125	·99687	1·0025	1·0082	1·0138	1·0195	1·0252
28	·95694	·96802	·97917	·99036	·99598	1·0016	1·0073	1·0129	1·0186	1·0243
29	·95614	·96722	·97835	·98954	·99515	1·0008	1·0064	1·0121	1·0177	1·0234
30	0·95540	0·96647	0·97759	0·98877	0·99438	1·0000	1·0056	1·0113	1·0170	1·0226
40	·95003	·96104	·97211	·98323	·98880	0·99440	1·0000	1·0056	1·0113	1·0169
60	·94471	·95566	·96667	·97773	·98328	·98884	0·99441	1·0000	1·0056	1·0112
120	·93943	·95032	·96128	·97228	·97780	·98333	·98887	0·99443	1·0000	1·0056
∞	·93418	·94503	·95593	·96687	·97236	·97787	·98339	·98891	0·99445	1·0000

$$F = \frac{s_1^2}{s_2^2} = \frac{\nu_2 S_1}{\nu_1 S_2}.$$

F Distribution: 25 per cent Points

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9
1	5·8285	7·5000	8·1999	8·5810	8·8198	8·9833	9·1021	9·1922	9·2631
2	2·5714	3·0000	3·1534	3·2320	3·2799	3·3121	3·3352	3·3526	3·3661
3	2·0239	2·2798	2·3555	2·3901	2·4095	2·4218	2·4302	2·4364	2·4410
4	1·8074	2·0000	2·0467	2·0642	2·0723	2·0766	2·0790	2·0805	2·0814
5	1·6925	1·8528	1·8843	1·8927	1·8947	1·8945	1·8935	1·8923	1·8911
6	1·6214	1·7622	1·7844	1·7872	1·7852	1·7821	1·7789	1·7760	1·7733
7	1·5732	1·7010	1·7169	1·7157	1·7111	1·7059	1·7011	1·6969	1·6931
8	1·5384	1·6569	1·6683	1·6642	1·6575	1·6508	1·6448	1·6396	1·6350
9	1·5121	1·6236	1·6315	1·6253	1·6170	1·6091	1·6022	1·5961	1·5909
10	1·4915	1·5975	1·6028	1·5949	1·5853	1·5765	1·5688	1·5621	1·5563
11	1·4749	1·5767	1·5798	1·5704	1·5598	1·5502	1·5418	1·5346	1·5284
12	1·4613	1·5595	1·5609	1·5503	1·5389	1·5286	1·5197	1·5120	1·5054
13	1·4500	1·5452	1·5451	1·5336	1·5214	1·5105	1·5011	1·4931	1·4861
14	1·4403	1·5331	1·5317	1·5194	1·5066	1·4952	1·4854	1·4770	1·4697
15	1·4321	1·5227	1·5202	1·5071	1·4938	1·4820	1·4718	1·4631	1·4556
16	1·4249	1·5137	1·5103	1·4965	1·4827	1·4705	1·4601	1·4511	1·4433
17	1·4186	1·5057	1·5015	1·4873	1·4730	1·4605	1·4497	1·4405	1·4325
18	1·4130	1·4988	1·4938	1·4790	1·4644	1·4516	1·4406	1·4312	1·4230
19	1·4081	1·4925	1·4870	1·4717	1·4568	1·4437	1·4325	1·4228	1·4145
20	1·4037	1·4870	1·4808	1·4652	1·4500	1·4366	1·4252	1·4153	1·4069
21	1·3997	1·4820	1·4753	1·4593	1·4438	1·4302	1·4186	1·4086	1·4000
22	1·3961	1·4774	1·4703	1·4540	1·4382	1·4244	1·4126	1·4025	1·3937
23	1·3928	1·4733	1·4657	1·4491	1·4331	1·4191	1·4072	1·3969	1·3880
24	1·3898	1·4695	1·4615	1·4447	1·4285	1·4143	1·4022	1·3918	1·3828
25	1·3870	1·4661	1·4577	1·4406	1·4242	1·4099	1·3976	1·3871	1·3780
26	1·3845	1·4629	1·4542	1·4368	1·4203	1·4058	1·3935	1·3828	1·3737
27	1·3822	1·4600	1·4510	1·4334	1·4166	1·4021	1·3896	1·3788	1·3696
28	1·3800	1·4572	1·4480	1·4302	1·4133	1·3986	1·3860	1·3752	1·3658
29	1·3780	1·4547	1·4452	1·4272	1·4102	1·3953	1·3826	1·3717	1·3623
30	1·3761	1·4524	1·4426	1·4244	1·4073	1·3923	1·3795	1·3685	1·3590
40	1·3626	1·4355	1·4239	1·4045	1·3863	1·3706	1·3571	1·3455	1·3354
60	1·3493	1·4188	1·4055	1·3848	1·3657	1·3491	1·3349	1·3226	1·3119
120	1·3362	1·4024	1·3873	1·3654	1·3453	1·3278	1·3128	1·2999	1·2886
∞	1·3233	1·3863	1·3694	1·3463	1·3251	1·3068	1·2910	1·2774	1·2654

This table gives the values of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}25$.

F DISTRIBUTION: 25 PER CENT POINTS

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	9·3202	9·4064	9·4934	9·5813	9·6255	9·6698	9·7144	9·7591	9·8041	9·8492
2	3·3770	3·3934	3·4098	3·4263	3·4345	3·4428	3·4511	3·4594	3·4677	3·4761
3	2·4447	2·4500	2·4552	2·4602	2·4626	2·4650	2·4674	2·4697	2·4720	2·4742
4	2·0820	2·0826	2·0829	2·0828	2·0827	2·0825	2·0821	2·0817	2·0812	2·0806
5	1·8899	1·8877	1·8851	1·8820	1·8802	1·8784	1·8763	1·8742	1·8719	1·8694
6	1·7708	1·7668	1·7621	1·7569	1·7540	1·7510	1·7477	1·7443	1·7407	1·7368
7	1·6898	1·6843	1·6781	1·6712	1·6675	1·6635	1·6593	1·6548	1·6502	1·6452
8	1·6310	1·6244	1·6170	1·6088	1·6043	1·5996	1·5945	1·5892	1·5836	1·5777
9	1·5863	1·5788	1·5705	1·5611	1·5560	1·5506	1·5450	1·5389	1·5325	1·5257
10	1·5513	1·5430	1·5338	1·5235	1·5179	1·5119	1·5056	1·4990	1·4919	1·4843
11	1·5230	1·5140	1·5041	1·4930	1·4869	1·4805	1·4737	1·4664	1·4587	1·4504
12	1·4996	1·4902	1·4796	1·4678	1·4613	1·4544	1·4471	1·4393	1·4310	1·4221
13	1·4801	1·4701	1·4590	1·4465	1·4397	1·4324	1·4247	1·4164	1·4075	1·3980
14	1·4634	1·4530	1·4414	1·4284	1·4212	1·4136	1·4055	1·3967	1·3874	1·3772
15	1·4491	1·4383	1·4263	1·4127	1·4052	1·3973	1·3888	1·3796	1·3698	1·3591
16	1·4366	1·4255	1·4130	1·3990	1·3913	1·3830	1·3742	1·3646	1·3543	1·3432
17	1·4256	1·4142	1·4014	1·3869	1·3790	1·3704	1·3613	1·3514	1·3406	1·3290
18	1·4159	1·4042	1·3911	1·3762	1·3680	1·3592	1·3497	1·3395	1·3284	1·3162
19	1·4073	1·3953	1·3819	1·3666	1·3582	1·3492	1·3394	1·3289	1·3174	1·3048
20	1·3995	1·3873	1·3736	1·3580	1·3494	1·3401	1·3301	1·3193	1·3074	1·2943
21	1·3925	1·3801	1·3661	1·3502	1·3414	1·3319	1·3217	1·3105	1·2983	1·2848
22	1·3861	1·3735	1·3593	1·3431	1·3341	1·3245	1·3140	1·3025	1·2900	1·2761
23	1·3803	1·3675	1·3531	1·3366	1·3275	1·3176	1·3069	1·2952	1·2824	1·2681
24	1·3750	1·3621	1·3474	1·3307	1·3214	1·3113	1·3004	1·2885	1·2754	1·2607
25	1·3701	1·3570	1·3422	1·3252	1·3158	1·3056	1·2945	1·2823	1·2689	1·2538
26	1·3656	1·3524	1·3374	1·3202	1·3106	1·3002	1·2889	1·2765	1·2628	1·2474
27	1·3615	1·3481	1·3329	1·3155	1·3058	1·2953	1·2838	1·2712	1·2572	1·2414
28	1·3576	1·3441	1·3288	1·3112	1·3013	1·2906	1·2790	1·2662	1·2519	1·2358
29	1·3541	1·3404	1·3249	1·3071	1·2971	1·2863	1·2745	1·2615	1·2470	1·2306
30	1·3507	1·3369	1·3213	1·3033	1·2933	1·2823	1·2703	1·2571	1·2424	1·2256
40	1·3266	1·3119	1·2952	1·2758	1·2649	1·2529	1·2397	1·2249	1·2080	1·1883
60	1·3026	1·2870	1·2691	1·2481	1·2361	1·2229	1·2081	1·1912	1·1715	1·1474
120	1·2787	1·2621	1·2428	1·2200	1·2068	1·1921	1·1752	1·1555	1·1314	1·0987
∞	1·2549	1·2371	1·2163	1·1914	1·1767	1·1600	1·1404	1·1164	1·0838	1·0000

$$F = \frac{s_1^2}{s_2^2} = \frac{\nu_2 S_1}{\nu_1 S_2}.$$

F Distribution: 10 per cent Points

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9
1	39·864	49·500	53·593	55·833	57·241	58·204	58·906	59·439	59·858
2	8·5263	9·0000	9·1618	9·2434	9·2926	9·3255	9·3491	9·3668	9·3805
3	5·5383	5·4624	5·3908	5·3427	5·3092	5·2847	5·2662	5·2517	5·2400
4	4·5448	4·3246	4·1908	4·1073	4·0506	4·0098	3·9790	3·9549	3·9357
5	4·0604	3·7797	3·6195	3·5202	3·4530	3·4045	3·3679	3·3393	3·3163
6	3·7760	3·4633	3·2888	3·1808	3·1075	3·0546	3·0145	2·9830	2·9577
7	3·5894	3·2574	3·0741	2·9605	2·8833	2·8274	2·7849	2·7516	2·7247
8	3·4579	3·1131	2·9238	2·8064	2·7265	2·6683	2·6241	2·5893	2·5612
9	3·3603	3·0065	2·8129	2·6927	2·6106	2·5509	2·5053	2·4694	2·4403
10	3·2850	2·9245	2·7277	2·6053	2·5216	2·4606	2·4140	2·3772	2·3473
11	3·2252	2·8595	2·6602	2·5362	2·4512	2·3891	2·3416	2·3040	2·2735
12	3·1765	2·8068	2·6055	2·4801	2·3940	2·3310	2·2828	2·2446	2·2135
13	3·1362	2·7632	2·5603	2·4337	2·3467	2·2830	2·2341	2·1953	2·1638
14	3·1022	2·7265	2·5222	2·3947	2·3069	2·2426	2·1931	2·1539	2·1220
15	3·0732	2·6952	2·4898	2·3614	2·2730	2·2081	2·1582	2·1185	2·0862
16	3·0481	2·6682	2·4618	2·3327	2·2438	2·1783	2·1280	2·0880	2·0553
17	3·0262	2·6446	2·4374	2·3077	2·2183	2·1524	2·1017	2·0613	2·0284
18	3·0070	2·6239	2·4160	2·2858	2·1958	2·1296	2·0785	2·0379	2·0047
19	2·9899	2·6056	2·3970	2·2663	2·1760	2·1094	2·0580	2·0171	1·9836
20	2·9747	2·5893	2·3801	2·2489	2·1582	2·0913	2·0397	1·9985	1·9649
21	2·9609	2·5746	2·3649	2·2333	2·1423	2·0751	2·0232	1·9819	1·9480
22	2·9486	2·5613	2·3512	2·2193	2·1279	2·0605	2·0084	1·9668	1·9327
23	2·9374	2·5493	2·3387	2·2065	2·1149	2·0472	1·9949	1·9531	1·9189
24	2·9271	2·5383	2·3274	2·1949	2·1030	2·0351	1·9826	1·9407	1·9063
25	2·9177	2·5283	2·3170	2·1843	2·0922	2·0241	1·9714	1·9292	1·8947
26	2·9091	2·5191	2·3075	2·1745	2·0822	2·0139	1·9610	1·9188	1·8841
27	2·9012	2·5106	2·2987	2·1655	2·0730	2·0045	1·9515	1·9091	1·8743
28	2·8939	2·5028	2·2906	2·1571	2·0645	1·9959	1·9427	1·9001	1·8652
29	2·8871	2·4955	2·2831	2·1494	2·0566	1·9878	1·9345	1·8918	1·8568
30	2·8807	2·4887	2·2761	2·1422	2·0492	1·9803	1·9269	1·8841	1·8490
40	2·8354	2·4404	2·2261	2·0909	1·9968	1·9269	1·8725	1·8289	1·7929
60	2·7914	2·3932	2·1774	2·0410	1·9457	1·8747	1·8194	1·7748	1·7380
120	2·7478	2·3473	2·1300	1·9923	1·8959	1·8238	1·7675	1·7220	1·6843
∞	2·7055	2·3026	2·0838	1·9449	1·8473	1·7741	1·7167	1·6702	1·6315

This table gives the values of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}10$.

F DISTRIBUTION: 10 PER CENT POINTS

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	60·195	60·705	61·220	61·740	62·002	62·265	62·529	62·794	63·061	63·328
2	9·3916	9·4081	9·4247	9·4413	9·4496	9·4579	9·4663	9·4746	9·4829	9·4913
3	5·2304	5·2156	5·2003	5·1845	5·1764	5·1681	5·1597	5·1512	5·1425	5·1337
4	3·9199	3·8955	3·8689	3·8443	3·8310	3·8174	3·8036	3·7896	3·7753	3·7607
5	3·2974	3·2682	3·2380	3·2067	3·1905	3·1741	3·1573	3·1402	3·1228	3·1050
6	2·9369	2·9047	2·8712	2·8363	2·8183	2·8000	2·7812	2·7620	2·7423	2·7222
7	2·7025	2·6681	2·6322	2·5947	2·5753	2·5555	2·5351	2·5142	2·4928	2·4708
8	2·5380	2·5020	2·4642	2·4246	2·4041	2·3830	2·3614	2·3391	2·3162	2·2926
9	2·4163	2·3789	2·3396	2·2983	2·2768	2·2547	2·2320	2·2085	2·1843	2·1592
10	2·3226	2·2841	2·2435	2·2007	2·1784	2·1554	2·1317	2·1072	2·0818	2·0554
11	2·2482	2·2087	2·1671	2·1230	2·1000	2·0762	2·0516	2·0261	1·9997	1·9721
12	2·1878	2·1474	2·1049	2·0597	2·0360	2·0115	1·9861	1·9597	1·9323	1·9036
13	2·1376	2·0966	2·0532	2·0070	1·9827	1·9576	1·9315	1·9043	1·8759	1·8462
14	2·0954	2·0537	2·0095	1·9625	1·9377	1·9119	1·8852	1·8572	1·8280	1·7973
15	2·0593	2·0171	1·9722	1·9243	1·8990	1·8728	1·8454	1·8168	1·7867	1·7551
16	2·0281	1·9854	1·9399	1·8913	1·8656	1·8388	1·8108	1·7816	1·7507	1·7182
17	2·0009	1·9577	1·9117	1·8624	1·8362	1·8090	1·7805	1·7506	1·7191	1·6856
18	1·9770	1·9333	1·8868	1·8368	1·8103	1·7827	1·7537	1·7232	1·6910	1·6567
19	1·9557	1·9117	1·8647	1·8142	1·7873	1·7592	1·7298	1·6988	1·6659	1·6308
20	1·9367	1·8924	1·8449	1·7938	1·7667	1·7382	1·7083	1·6768	1·6433	1·6074
21	1·9197	1·8750	1·8272	1·7756	1·7481	1·7193	1·6890	1·6569	1·6228	1·5862
22	1·9043	1·8593	1·8111	1·7590	1·7312	1·7021	1·6714	1·6389	1·6042	1·5668
23	1·8903	1·8450	1·7964	1·7439	1·7159	1·6864	1·6554	1·6224	1·5871	1·5490
24	1·8775	1·8319	1·7831	1·7302	1·7019	1·6721	1·6407	1·6073	1·5715	1·5327
25	1·8658	1·8200	1·7708	1·7175	1·6890	1·6589	1·6272	1·5934	1·5570	1·5176
26	1·8550	1·8090	1·7596	1·7059	1·6771	1·6468	1·6147	1·5805	1·5437	1·5036
27	1·8451	1·7989	1·7492	1·6951	1·6662	1·6356	1·6032	1·5686	1·5313	1 4906
28	1·8359	1·7895	1·7395	1·6852	1·6560	1·6252	1·5925	1·5575	1·5198	1·4784
29	1·8274	1·7808	1·7306	1·6759	1·6465	1·6155	1·5825	1·5472	1·5090	1·4670
30	1·8195	1·7727	1·7223	1·6673	1·6377	1·6065	1·5732	1·5376	1·4989	1·4564
40	1·7627	1·7146	1·6624	1·6052	1·5741	1·5411	1·5056	1·4672	1·4248	1·3769
60	1·7070	1·6574	1·6034	1·5435	1·5107	1·4755	1·4373	1·3952	1·3476	1·2915
120	1·6524	1·6012	1·5450	1·4821	1·4472	1·4094	1·3676	1·3203	1·2646	1·1926
∞	1·5987	1·5458	1·4871	1·4206	1·3832	1·3419	1·2951	1·2400	1·1686	1·0000

$$F = \frac{s_1^2}{s_2^2} = \frac{\nu_2 S_1}{\nu_1 S_2}.$$

F Distribution: 5 per cent Points

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9
1	161·45	199·50	215·71	224·58	230·16	233·99	236·77	238·88	240·54
2	18·513	19·000	19·164	19·247	19·296	19·330	19·353	19·371	19·385
3	10·128	9·5521	9·2766	9·1172	9·0135	8·9406	8·8868	8·8452	8·8123
4	7·7086	6·9443	6·5914	6·3883	6·2560	6·1631	6·0942	6·0410	5·9988
5	6·6079	5·7861	5·4095	5·1922	5·0503	4·9503	4·8759	4·8183	4·7725
6	5·9874	5·1433	4·7571	4·5337	4·3874	4·2839	4·2066	4·1468	4·0990
7	5·5914	4·7374	4·3468	4·1203	3·9715	3·8660	3·7870	3·7257	3·6767
8	5·3177	4·4590	4·0662	3·8378	3·6875	3·5806	3·5005	3·4381	3·3881
9	5·1174	4·2565	3·8626	3·6331	3·4817	3·3738	3·2927	3·2296	3·1789
10	4·9646	4·1028	3·7083	3·4780	3·3258	3·2172	3·1355	3·0717	3·0204
11	4·8443	3·9823	3·5874	3·3567	3·2039	3·0946	3·0123	2·9480	2·8962
12	4·7472	3·8853	3·4903	3·2592	3·1059	2·9961	2·9134	2·8486	2·7964
13	4·6672	3·8056	3·4105	3·1791	3·0254	2·9153	2·8321	2·7669	2·7144
14	4·6001	3·7389	3·3439	3·1122	2·9582	2·8477	2·7642	2·6987	2·6458
15	4·5431	3·6823	3·2874	3·0556	2·9013	2·7905	2·7066	2·6408	2·5876
16	4·4940	3·6337	3·2389	3·0069	2·8524	2·7413	2·6572	2·5911	2·5377
17	4·4513	3·5915	3·1968	2·9647	2·8100	2·6987	2·6143	2·5480	2·4943
18	4·4139	3·5546	3·1599	2·9277	2·7729	2·6613	2·5767	2·5102	2·4563
19	4·3808	3·5219	3·1274	2·8951	2·7401	2·6283	2·5435	2·4768	2·4227
20	4·3513	3·4928	3·0984	2·8661	2·7109	2·5990	2·5140	2·4471	2·3928
21	4·3248	3·4668	3·0725	2·8401	2·6848	2·5727	2·4876	2·4205	2·3661
22	4·3009	3·4434	3·0491	2·8167	2·6613	2·5491	2·4638	2·3965	2·3419
23	4·2793	3·4221	3·0280	2·7955	2·6400	2·5277	2·4422	2·3748	2·3201
24	4·2597	3·4028	3·0088	2·7763	2·6207	2·5082	2·4226	2·3551	2·3002
25	4·2417	3·3852	2·9912	2·7587	2·6030	2·4904	2·4047	2·3371	2·2821
26	4·2252	3·3690	2·9751	2·7426	2·5868	2·4741	2·3883	2·3205	2·2655
27	4·2100	3·3541	2·9604	2·7278	2·5719	2·4591	2·3732	2·3053	2·2501
28	4·1960	3·3404	2·9467	2·7141	2·5581	2·4453	2·3593	2·2913	2·2360
29	4·1830	3·3277	2·9340	2·7014	2·5454	2·4324	2·3463	2·2782	2·2229
30	4·1709	3·3158	2·9223	2·6896	2·5336	2·4205	2·3343	2·2662	2·2107
40	4·0848	3·2317	2·8387	2·6060	2·4495	2·3359	2·2490	2·1802	2·1240
60	4·0012	3·1504	2·7581	2·5252	2·3683	2·2540	2·1665	2·0970	2·0401
120	3·9201	3·0718	2·6802	2·4472	2·2900	2·1750	2·0867	2·0164	1·9588
∞	3·8415	2·9957	2·6049	2·3719	2·2141	2·0986	2·0096	1·9384	1·8799

This table gives the values of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}05$.

F DISTRIBUTION: 5 PER CENT POINTS

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	241·88	243·91	245·95	248·01	249·05	250·09	251·14	252·20	253·25	254·32
2	19·396	19·413	19·429	19·446	19·454	19·462	19·471	19·479	19·487	19·496
3	8·7855	8·7446	8·7029	8·6602	8·6385	8·6166	8·5944	8·5720	8·5494	8·5265
4	5·9644	5·9117	5·8578	5·8025	5·7744	5·7459	5·7170	5·6878	5·6581	5·6281
5	4·7351	4·6777	4·6188	4·5581	4·5272	4·4957	4·4638	4·4314	4·3984	4·3650
6	4·0600	3·9999	3·9381	3·8742	3·8415	3·8082	3·7743	3·7398	3·7047	3·6688
7	3·6365	3·5747	3·5108	3·4445	3·4105	3·3758	3·3404	3·3043	3·2674	3·2298
8	3·3472	3·2840	3·2184	3·1503	3·1152	3·0794	3·0428	3·0053	2·9669	2·9276
9	3·1373	3·0729	3·0061	2·9365	2·9005	2·8637	2·8259	2·7872	2·7475	2·7067
10	2·9782	2·9130	2·8450	2·7740	2·7372	2·6996	2·6609	2·6211	2·5801	2·5379
11	2·8536	2·7876	2·7186	2·6464	2·6090	2·5705	2·5309	2·4901	2·4480	2·4045
12	2·7534	2·6866	2·6169	2·5436	2·5055	2·4663	2·4259	2·3842	2·3410	2·2962
13	2·6710	2·6037	2·5331	2·4589	2·4202	2·3803	2·3392	2·2966	2·2524	2·2064
14	2·6021	2·5342	2·4630	2·3879	2·3487	2·3082	2·2664	2·2230	2·1778	2·1307
15	2·5437	2·4753	2·4035	2·3275	2·2878	2·2468	2·2043	2·1601	2·1141	2·0658
16	2·4935	2·4247	2·3522	2·2756	2·2354	2·1938	2·1507	2·1058	2·0589	2·0096
17	2·4499	2·3807	2·3077	2·2304	2·1898	2·1477	2·1040	2·0584	2·0107	1·9604
18	2·4117	2·3421	2·2686	2·1906	2·1497	2·1071	2·0629	2·0166	1·9681	1·9168
19	2·3779	2·3080	2·2341	2·1555	2·1141	2·0712	2·0264	1·9796	1·9302	1·8780
20	2·3479	2·2776	2·2033	2·1242	2·0825	2·0391	1·9938	1·9464	1·8963	1·8432
21	2·3210	2·2504	2·1757	2·0960	2·0540	2·0102	1·9645	1·9165	1·8657	1·8117
22	2·2967	2·2258	2·1508	2·0707	2·0283	1·9842	1·9380	1·8895	1·8380	1·7831
23	2·2747	2·2036	2·1282	2·0476	2·0050	1·9605	1·9139	1·8649	1·8128	1·7570
24	2·2547	2·1834	2·1077	2·0267	1·9838	1·9390	1·8920	1·8424	1·7897	1·7331
25	2·2365	2·1649	2·0889	2·0075	1·9643	1·9192	1·8718	1·8217	1·7684	1·7110
26	2·2197	2·1479	2·0716	1·9898	1·9464	1·9010	1·8533	1·8027	1·7488	1·6906
27	2·2043	2·1323	2·0558	1·9736	1·9299	1·8842	1·8361	1·7851	1·7307	1·6717
28	2·1900	2·1179	2·0411	1·9586	1·9147	1·8687	1·8203	1·7689	1·7138	1·6541
29	2·1768	2·1045	2·0275	1·9446	1·9005	1·8543	1·8055	1·7537	1·6981	1·6377
30	2·1646	2·0921	2·0148	1·9317	1·8874	1·8409	1·7918	1·7396	1·6835	1·6223
40	2·0772	2·0035	1·9245	1·8389	1·7929	1·7444	1·6928	1·6373	1·5766	1·5089
60	1·9926	1·9174	1·8364	1·7480	1·7001	1·6491	1·5943	1·5343	1·4673	1·3893
120	1·9105	1·8337	1·7505	1·6587	1·6084	1·5543	1·4952	1·4290	1·3519	1·2539
∞	1·8307	1·7522	1·6664	1·5705	1·5173	1·4591	1·3940	1·3180	1·2214	1·0000

$$F = \frac{s_1^2}{s_2^2} = \frac{\nu_2 S_1}{\nu_1 S_2}.$$

F Distribution: 2·5 per cent Points

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9
1	647·79	799·50	864·16	899·58	921·85	937·11	948·22	956·66	963·28
2	38·506	39·000	39·165	39·248	39·298	39·331	39·355	39·373	39·387
3	17·443	16·044	15·439	15·101	14·885	14·735	14·624	14·540	14·473
4	12·218	10·649	9·9792	9·6045	9·3645	9·1973	9·0741	8·9796	8·9047
5	10·007	8·4336	7·7636	7·3879	7·1464	6·9777	6·8531	6·7572	6·6810
6	8·8131	7·2598	6·5988	6·2272	5·9876	5·8197	5·6955	5·5996	5·5234
7	8·0727	6·5415	5·8898	5·5226	5·2852	5·1186	4·9949	4·8994	4·8232
8	7·5709	6·0595	5·4160	5·0526	4·8173	4·6517	4·5286	4·4332	4·3572
9	7·2093	5·7147	5·0781	4·7181	4·4844	4·3197	4·1971	4·1020	4·0260
10	6·9367	5·4564	4·8256	4·4683	4·2361	4·0721	3·9498	3·8549	3·7790
11	6·7241	5·2559	4·6300	4·2751	4·0440	3·8807	3·7586	3·6638	3·5879
12	6·5538	5·0959	4·4742	4·1212	3·8911	3·7283	3·6065	3·5118	3·4358
13	6·4143	4·9653	4·3472	3·9959	3·7667	3·6043	3·4827	3·3880	3·3120
14	6·2979	4·8567	4·2417	3·8919	3·6634	3·5014	3·3799	3·2853	3·2093
15	6·1995	4·7650	4·1528	3·8043	3·5764	3·4147	3·2934	3·1987	3·1227
16	6·1151	4·6867	4·0768	3·7294	3·5021	3·3406	3·2194	3·1248	3·0488
17	6·0420	4·6189	4·0112	3·6648	3·4379	3·2767	3·1556	3·0610	2·9849
18	5·9781	4·5597	3·9539	3·6083	3·3820	3·2209	3·0999	3·0053	2·9291
19	5·9216	4·5075	3·9034	3·5587	3·3327	3·1718	3·0509	2·9563	2·8800
20	5·8715	4·4613	3·8587	3·5147	3·2891	3·1283	3·0074	2·9128	2·8365
21	5·8266	4·4199	3·8188	3·4754	3·2501	3·0895	2·9686	2·8740	2·7977
22	5·7863	4·3828	3·7829	3·4401	3·2151	3·0546	2·9338	2·8392	2·7628
23	5·7498	4·3492	3·7505	3·4083	3·1835	3·0232	2·9024	2·8077	2·7313
24	5·7167	4·3187	3·7211	3·3794	3·1548	2·9946	2·8738	2·7791	2·7027
25	5·6864	4·2909	3·6943	3·3530	3·1287	2·9685	2·8478	2·7531	2·6766
26	5·6586	4·2655	3·6697	3·3289	3·1048	2·9447	2·8240	2·7293	2·6528
27	5·6331	4·2421	3·6472	3·3067	3·0828	2·9228	2·8021	2·7074	2·6309
28	5·6096	4·2205	3·6264	3·2863	3·0625	2·9027	2·7820	2·6872	2·6106
29	5·5878	4·2006	3·6072	3·2674	3·0438	2·8840	2·7633	2·6686	2·5919
30	5·5675	4·1821	3·5894	3·2499	3·0265	2·8667	2·7460	2·6513	2·5746
40	5·4239	4·0510	3·4633	3·1261	2·9037	2·7444	2·6238	2·5289	2·4519
60	5·2857	3·9253	3·3425	3·0077	2·7863	2·6274	2·5068	2·4117	2·3344
120	5·1524	3·8046	3·2270	2·8943	2·6740	2·5154	2·3948	2·2994	2·2217
∞	5·0239	3·6889	3·1161	2·7858	2·5665	2·4082	2·2875	2·1918	2·1136

This table gives the values of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}025$.

F DISTRIBUTION: 2·5 PER CENT POINTS

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	968·63	976·71	984·87	993·10	997·25	1001·4	1005·6	1009·8	1014·0	1018·3
2	39·398	39·415	39·431	39·448	39·456	39·465	39·473	39·481	39·490	39·498
3	14·419	14·337	14·253	14·167	14·124	14·081	14·037	13·992	13·947	13·902
4	8·8439	8·7512	8·6565	8·5599	8·5109	8·4613	8·4111	8·3604	8·3092	8·2573
5	6·6192	6·5246	6·4277	6·3285	6·2780	6·2269	6·1751	6·1225	6·0693	6·0153
6	5·4613	5·3662	5·2687	5·1684	5·1172	5·0652	5·0125	4·9589	4·9045	4·8491
7	4·7611	4·6658	4·5678	4·4667	4·4150	4·3624	4·3089	4·2544	4·1989	4·1423
8	4·2951	4·1997	4·1012	3·9995	3·9472	3·8940	3·8398	3·7844	3·7279	3·6702
9	3·9639	3·8682	3·7694	3·6669	3·6142	3·5604	3·5055	3·4493	3·3918	3·3329
10	3·7168	3·6209	3·5217	3·4186	3·3654	3·3110	3·2554	3·1984	3·1399	3·0798
11	3·5257	3·4296	3·3299	3·2261	3·1725	3·1176	3·0613	3·0035	2·9441	2·8828
12	3·3736	3·2773	3·1772	3·0728	3·0187	2·9633	2·9063	2·8478	2·7874	2·7249
13	3·2497	3·1532	3·0527	2·9477	2·8932	2·8373	2·7797	2·7204	2·6590	2·5955
14	3·1469	3·0501	2·9493	2·8437	2·7888	2·7324	2·6742	2·6142	2·5519	2·4872
15	3·0602	2·9633	2·8621	2·7559	2·7006	2·6437	2·5850	2·5242	2·4611	2·3953
16	2·9862	2·8890	2·7875	2·6808	2·6252	2·5678	2·5085	2·4471	2·3831	2·3163
17	2·9222	2·8249	2·7230	2·6158	2·5598	2·5021	2·4422	2·3801	2·3153	2·2474
18	2·8664	2·7689	2·6667	2·5590	2·5027	2·4445	2·3842	2·3214	2·2558	2·1869
19	2·8173	2·7196	2·6171	2·5089	2·4523	2·3937	2·3329	2·2695	2·2032	2·1333
20	2·7737	2·6758	2·5731	2·4645	2·4076	2·3486	2·2873	2·2234	2·1562	2·0853
21	2·7348	2·6368	2·5338	2·4247	2·3675	2·3082	2·2465	2·1819	2·1141	2·0422
22	2·6998	2·6017	2·4984	2·3890	2·3315	2·2718	2·2097	2·1446	2·0760	2·0032
23	2·6682	2·5699	2·4665	2·3567	2·2989	2·2389	2·1763	2·1107	2·0415	1·9677
24	2·6396	2·5412	2·4374	2·3273	2·2693	2·2090	2·1460	2·0799	2·0099	1·9353
25	2·6135	2·5149	2·4110	2·3005	2·2422	2·1816	2·1183	2·0517	1·9811	1·9055
26	2·5895	2·4909	2·3867	2·2759	2·2174	2·1565	2·0928	2·0257	1·9545	1·8781
27	2·5676	2·4688	2·3644	2·2533	2·1946	2·1334	2·0693	2·0018	1·9299	1·8527
28	2·5473	2·4484	2·3438	2·2324	2·1735	2·1121	2·0477	1·9796	1·9072	1·8291
29	2·5286	2·4295	2·3248	2·2131	2·1540	2·0923	2·0276	1·9591	1·8861	1·8072
30	2·5112	2·4120	2·3072	2·1952	2·1359	2·0739	2·0089	1·9400	1·8664	1·7867
40	2·3882	2·2882	2·1819	2·0677	2·0069	1·9429	1·8752	1·8028	1·7242	1·6371
60	2·2702	2·1692	2·0613	1·9445	1·8817	1·8152	1·7440	1·6668	1·5810	1·4822
120	2·1570	2·0548	1·9450	1·8249	1·7597	1·6899	1·6141	1·5299	1·4327	1·3104
∞	2·0483	1·9447	1·8326	1·7085	1·6402	1·5660	1·4835	1·3883	1·2684	1·0000

$$F = \frac{s_1^2}{s_2^2} = \frac{\nu_2 S_1}{\nu_1 S_2}.$$

F DISTRIBUTION: 1 PER CENT POINTS

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9
1	4052·2	4999·5	5403·3	5624·6	5763·7	5859·0	5928·3	5981·6	6022·5
2	98·503	99·000	99·166	99·249	99·299	99·332	99·356	99·374	99·388
3	34·116	30·817	29·457	28·710	28·237	27·911	27·672	27·489	27·345
4	21·198	18·000	16·694	15·977	15·522	15·207	14·976	14·799	14·659
5	16·258	13·274	12·060	11·392	10·967	10·672	10·456	10·289	10·158
6	13·745	10·925	9·7795	9·1483	8·7459	8·4661	8·2600	8·1016	7·9761
7	12·246	9·5466	8·4513	7·8467	7·4604	7·1914	6·9928	6·8401	6·7188
8	11·259	8·6491	7·5910	7·0060	6·6318	6·3707	6·1776	6·0289	5·9106
9	10·561	8·0215	6·9919	6·4221	6·0569	5·8018	5·6129	5·4671	5·3511
10	10·044	7·5594	6·5523	5·9943	5·6363	5·3858	5·2001	5·0567	4·9424
11	9·6460	7·2057	6·2167	5·6683	5·3160	5·0692	4·8861	4·7445	4·6315
12	9·3302	6·9266	5·9526	5·4119	5·0643	4·8206	4·6395	4·4994	4·3875
13	9·0738	6·7010	5·7394	5·2053	4·8616	4·6204	4·4410	4·3021	4·1911
14	8·8616	6·5149	5·5639	5·0354	4·6950	4·4558	4·2779	4·1399	4·0297
15	8·6831	6·3589	5·4170	4·8932	4·5556	4·3183	4·1415	4·0045	3·8948
16	8·5310	6·2262	5·2922	4·7726	4·4374	4·2016	4·0259	3·8896	3·7804
17	8·3997	6·1121	5·1850	4·6690	4·3359	4·1015	3·9267	3·7910	3·6822
18	8·2854	6·0129	5·0919	4·5790	4·2479	4·0146	3·8406	3·7054	3·5971
19	8·1850	5·9259	5·0103	4·5003	4·1708	3·9386	3·7653	3·6305	3·5225
20	8·0960	5·8489	4·9382	4·4307	4·1027	3·8714	3·6987	3·5644	3·4567
21	8·0166	5·7804	4·8740	4·3688	4·0421	3·8117	3·6396	3·5056	3·3981
22	7·9454	5·7190	4·8166	4·3134	3·9880	3·7583	3·5867	3·4530	3·3458
23	7·8811	5·6637	4·7649	4·2635	3·9392	3·7102	3·5390	3·4057	3·2986
24	7·8229	5·6136	4·7181	4·2184	3·8951	3·6667	3·4959	3·3629	3·2560
25	7·7698	5·5680	4·6755	4·1774	3·8550	3·6272	3·4568	3·3239	3·2172
26	7·7213	5·5263	4·6366	4·1400	3·8183	3·5911	3·4210	3·2884	3·1818
27	7·6767	5·4881	4·6009	4·1056	3·7848	3·5580	3·3882	3·2558	3·1494
28	7·6356	5·4529	4·5681	4·0740	3·7539	3·5276	3·3581	3·2259	3·1195
29	7·5976	5·4205	4·5378	4·0449	3·7254	3·4995	3·3302	3·1982	3·0920
30	7·5625	5·3904	4·5097	4·0179	3·6990	3·4735	3·3045	3·1726	3·0665
40	7·3141	5·1785	4·3126	3·8283	3·5138	3·2910	3·1238	2·9930	2·8876
60	7·0771	4·9774	4·1259	3·6491	3·3389	3·1187	2·9530	2·8233	2·7185
120	6·8510	4·7865	3·9493	3·4796	3·1735	2·9559	2·7918	2·6629	2·5586
∞	6·6349	4·6052	3·7816	3·3192	3·0173	2·8020	2·6393	2·5113	2·4073

This table gives the values of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}01$.

F DISTRIBUTION: 1 PER CENT POINTS

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	6055·8	6106·3	6157·3	6208·7	6234·6	6260·7	6286·8	6313·0	6339·4	6366·0
2	99·399	99·416	99·432	99·449	99·458	99·466	99·474	99·483	99·491	99·501
3	27·229	27·052	26·872	26·690	26·598	26·505	26·411	26·316	26·221	26·125
4	14·546	14·374	14·198	14·020	13·929	13·838	13·745	13·652	13·558	13·463
5	10·051	9·8883	9·7222	9·5527	9·4665	9·3793	9·2912	9·2020	9·1118	9·0204
6	7·8741	7·7183	7·5590	7·3958	7·3127	7·2285	7·1432	7·0568	6·9690	6·8801
7	6·6201	6·4691	6·3143	6·1554	6·0743	5·9921	5·9084	5·8236	5·7372	5·6495
8	5·8143	5·6668	5·5151	5·3591	5·2793	5·1981	5·1156	5·0316	4·9460	4·8588
9	5·2565	5·1114	4·9621	4·8080	4·7290	4·6486	4·5667	4·4831	4·3978	4·3105
10	4·8492	4·7059	4·5582	4·4054	4·3269	4·2469	4·1653	4·0819	3·9965	3·9090
11	4·5393	4·3974	4·2509	4·0990	4·0209	3·9411	3·8596	3·7761	3·6904	3·6025
12	4·2961	4·1553	4·0096	3·8584	3·7805	3·7008	3·6192	3·5355	3·4494	3·3608
13	4·1003	3·9603	3·8154	3·6646	3·5868	3·5070	3·4253	3·3413	3·2548	3·1654
14	3·9394	3·8001	3·6557	3·5052	3·4274	3·3476	3·2656	3·1813	3·0942	3·0040
15	3·8049	3·6662	3·5222	3·3719	3·2940	3·2141	3·1319	3·0471	2·9595	2·8684
16	3·6909	3·5527	3·4089	3·2588	3·1808	3·1007	3·0182	2·9330	2·8447	2·7528
17	3·5931	3·4552	3·3117	3·1615	3·0835	3·0032	2·9205	2·8348	2·7459	2·6530
18	3·5082	3·3706	3·2273	3·0771	2·9990	2·9185	2·8354	2·7493	2·6597	2·5660
19	3·4338	3·2965	3·1533	3·0031	2·9249	2·8442	2·7608	2·6742	2·5839	2·4893
20	3·3682	3·2311	3·0880	2·9377	2·8594	2·7785	2·6947	2·6077	2·5168	2·4212
21	3·3098	3·1729	3·0299	2·8796	2·8011	2·7200	2·6359	2·5484	2·4568	2·3603
22	3·2576	3·1209	2·9780	2·8274	2·7488	2·6675	2·5831	2·4951	2·4029	2·3055
23	3·2106	3·0740	2·9311	2·7805	2·7017	2·6202	2·5355	2·4471	2·3542	2·2559
24	3·1681	3·0316	2·8887	2·7380	2·6591	2·5773	2·4923	2·4035	2·3099	2·2107
25	3·1294	2·9931	2·8502	2·6993	2·6203	2·5383	2·4530	2·3637	2·2695	2·1694
26	3·0941	2·9579	2·8150	2·6640	2·5848	2·5026	2·4170	2·3273	2·2325	2·1315
27	3·0618	2·9256	2·7827	2·6316	2·5522	2·4699	2·3840	2·2938	2·1984	2·0965
28	3·0320	2·8959	2·7530	2·6017	2·5223	2·4397	2·3535	2·2629	2·1670	2·0642
29	3·0045	2·8685	2·7256	2·5742	2·4946	2·4118	2·3253	2·2344	2·1378	2·0342
30	2·9791	2·8431	2·7002	2·5487	2·4689	2·3860	2·2992	2·2079	2·1107	2·0062
40	2·8005	2·6648	2·5216	2·3689	2·2880	2·2034	2·1142	2·0194	1·9172	1·8047
60	2·6318	2·4961	2·3523	2·1978	2·1154	2·0285	1·9360	1·8363	1·7263	1·6006
120	2·4721	2·3363	2·1915	2·0346	1·9500	1·8600	1·7628	1·6557	1·5330	1·3805
∞	2·3209	2·1848	2·0385	1·8783	1·7908	1·6964	1·5923	1·4730	1·3246	1·0000

$$F = \frac{s_1^2}{s_2^2} = \frac{\nu_2 S_1}{\nu_1 S_2}.$$

F Distribution: 0·5 per cent Points

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9
1	16211	20000	21615	22500	23056	23437	23715	23925	24091
2	198·50	199·00	199·17	199·25	199·30	199·33	199·36	199·37	199·39
3	55·552	49·799	47·467	46·195	45·392	44·838	44·434	44·126	43·882
4	31·333	26·284	24·259	23·155	22·456	21·975	21·622	21·352	21·139
5	22·785	18·314	16·530	15·556	14·940	14·513	14·200	13·961	13·772
6	18·635	14·544	12·917	12·028	11·464	11·073	10·786	10·566	10·391
7	16·236	12·404	10·882	10·050	9·5221	9·1554	8·8854	8·6781	8·5138
8	14·688	11·042	9·5965	8·8051	8·3018	7·9520	7·6942	7·4960	7·3386
9	13·614	10·107	8·7171	7·9559	7·4711	7·1338	6·8849	6·6933	6·5411
10	12·826	9·4270	8·0807	7·3428	6·8723	6·5446	6·3025	6·1159	5·9676
11	12·226	8·9122	7·6004	6·8809	6·4217	6·1015	5·8648	5·6821	5·5368
12	11·754	8·5096	7·2258	6·5211	6·0711	5·7570	5·5245	5·3451	5·2021
13	11·374	8·1865	6·9257	6·2335	5·7910	5·4819	5·2529	5·0761	4·9351
14	11·060	7·9217	6·6803	5·9984	5·5623	5·2574	5·0313	4·8566	4·7173
15	10·798	7·7008	6·4760	5·8029	5·3721	5·0708	4·8473	4·6743	4·5364
16	10·575	7·5138	6·3034	5·6378	5·2117	4·9134	4·6920	4·5207	4·3838
17	10·384	7·3536	6·1556	5·4967	5·0746	4·7789	4·5594	4·3893	4·2535
18	10·218	7·2148	6·0277	5·3746	4·9560	4·6627	4·4448	4·2759	4·1410
19	10·073	7·0935	5·9161	5·2681	4·8526	4·5614	4·3448	4·1770	4·0428
20	9·9439	6·9865	5·8177	5·1743	4·7616	4·4721	4·2569	4·0900	3·9564
21	9·8295	6·8914	5·7304	5·0911	4·6808	4·3931	4·1789	4·0128	3·8799
22	9·7271	6·8064	5·6524	5·0168	4·6088	4·3225	4·1094	3·9440	3·8116
23	9·6348	6·7300	5·5823	4·9500	4·5441	4·2591	4·0469	3·8822	3·7502
24	9·5513	6·6610	5·5190	4·8898	4·4857	4·2019	3·9905	3·8264	3·6949
25	9·4753	6·5982	5·4615	4·8351	4·4327	4·1500	3·9394	3·7758	3·6447
26	9·4059	6·5409	5·4091	4·7852	4·3844	4·1027	3·8928	3·7297	3·5989
27	9·3423	6·4885	5·3611	4·7396	4·3402	4·0594	3·8501	3·6875	3·5571
28	9·2838	6·4403	5·3170	4·6977	4·2996	4·0197	3·8110	3·6487	3·5186
29	9·2297	6·3958	5·2764	4·6591	4·2622	3·9830	3·7749	3·6130	3·4832
30	9·1797	6·3547	5·2388	4·6233	4·2276	3·9492	3·7416	3·5801	3·4505
40	8·8278	6·0664	4·9759	4·3738	3·9860	3·7129	3·5088	3·3498	3·2220
60	8·4946	5·7950	4·7290	4·1399	3·7600	3·4918	3·2911	3·1344	3·0083
120	8·1790	5·5393	4·4973	3·9207	3·5482	3·2849	3·0874	2·9330	2·8083
∞	7·8794	5·2983	4·2794	3·7151	3·3499	3·0913	2·8968	2·7444	2·6210

This table gives the values of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}005$.

F DISTRIBUTION: 0·5 PER CENT POINTS

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	24224	24426	24630	24836	24940	25044	25148	25253	25359	25465
2	199·40	199·42	199·43	199·45	199·46	199·47	199·47	199·48	199·49	199·51
3	43·686	43·387	43·085	42·778	42·622	42·466	42·308	42·149	41·989	41·829
4	20·967	20·705	20·438	20·167	20·030	19·892	19·752	19·611	19·468	19·325
5	13·618	13·384	13·146	12·903	12·780	12·656	12·530	12·402	12·274	12·144
6	10·250	10·034	9·8140	9·5888	9·4741	9·3583	9·2408	9·1219	9·0015	8·8793
7	8·3803	8·1764	7·9678	7·7540	7·6450	7·5345	7·4225	7·3088	7·1933	7·0760
8	7·2107	7·0149	6·8143	6·6082	6·5029	6·3961	6·2875	6·1772	6·0649	5·9505
9	6·4171	6·2274	6·0325	5·8318	5·7292	5·6248	5·5186	5·4104	5·3001	5·1875
10	5·8467	5·6613	5·4707	5·2740	5·1732	5·0705	4·9659	4·8592	4·7501	4·6385
11	5·4182	5·2363	5·0489	4·8552	4·7557	4·6543	4·5508	4·4450	4·3367	4·2256
12	5·0855	4·9063	4·7214	4·5299	4·4315	4·3309	4·2282	4·1229	4·0149	3·9039
13	4·8199	4·6429	4·4600	4·2703	4·1726	4·0727	3·9704	3·8655	3·7577	3·6465
14	4·6034	4·4281	4·2468	4·0585	3·9614	3·8619	3·7600	3·6553	3·5473	3·4359
15	4·4236	4·2498	4·0698	3·8826	3·7859	3·6867	3·5850	3·4803	3·3722	3·2602
16	4·2719	4·0994	3·9205	3·7342	3·6378	3·5388	3·4372	3·3324	3·2240	3·1115
17	4·1423	3·9709	3·7929	3·6073	3·5112	3·4124	3·3107	3·2058	3·0971	2·9839
18	4·0305	3·8599	3·6827	3·4977	3·4017	3·3030	3·2014	3·0962	2·9871	2·8732
19	3·9329	3·7631	3·5866	3·4020	3·3062	3·2075	3·1058	3·0004	2·8908	2·7762
20	3·8470	3·6779	3·5020	3·3178	3·2220	3·1234	3·0215	2·9159	2·8058	2·6904
21	3·7709	3·6024	3·4270	3·2431	3·1474	3·0488	2·9467	2·8408	2·7302	2·6140
22	3·7030	3·5350	3·3600	3·1764	3·0807	2·9821	2·8799	2·7736	2·6625	2·5455
23	3·6420	3·4745	3·2999	3·1165	3·0208	2·9221	2·8198	2·7132	2·6016	2·4837
24	3·5870	3·4199	3·2456	3·0624	2·9667	2·8679	2·7654	2·6585	2·5463	2·4276
25	3·5370	3·3704	3·1963	3·0133	2·9176	2·8187	2·7160	2·6088	2·4960	2·3765
26	3·4916	3·3252	3·1515	2·9685	2·8728	2·7738	2·6709	2·5633	2·4501	2·3297
27	3·4499	3·2839	3·1104	2·9275	2·8318	2·7327	2·6296	2·5217	2·4078	2·2867
28	3·4117	3·2460	3·0727	2·8899	2·7941	2·6949	2·5916	2·4834	2·3689	2·2469
29	3·3765	3·2111	3·0379	2·8551	2·7594	2·6601	2·5565	2·4479	2·3330	2·2102
30	3·3440	3·1787	3·0057	2·8230	2·7272	2·6278	2·5241	2·4151	2·2997	2·1760
40	3·1167	2·9531	2·7811	2·5984	2·5020	2·4015	2·2958	2·1838	2·0635	1·9318
60	2·9042	2·7419	2·5705	2·3872	2·2898	2·1874	2·0789	1·9622	1·8341	1·6885
120	2·7052	2·5439	2·3727	2·1881	2·0890	1·9839	1·8709	1·7469	1·6055	1·4311
∞	2·5188	2·3583	2·1868	1·9998	1·8983	1·7891	1·6691	1·5325	1·3637	1·0000

$$F = \frac{s_1^2}{s_2^2} = \frac{\nu_2 S_1}{\nu_1 S_2}.$$

The accompanying tables give seven upper percentage points for F, that is they give the roots of the equation

$$I_F(\nu_1, \nu_2) = \int_F^\infty f(F)\, dF \qquad (10)$$

for $100I_F(\nu_1, \nu_2) = 50, 25, 10, 5, 2{\cdot}5, 1$ and $0{\cdot}5$,

and for $\nu_1 = 1(1)10, 12, 15, 20, 24, 30, 40, 60, 120$ and ∞

$\nu_2 = 1(1)30, 40, 60, 120$ and ∞.

To obtain the lower percentage points, that is the roots of

$$I'_F(\nu_1, \nu_2) = \int_0^F f(F)\, dF = I_{1/F}(\nu_2, \nu_1),$$

it is only necessary to interchange the values of ν_1 and ν_2 in entering the tables and to take for F the reciprocal of the value so obtained. For example if $\nu_1 = 12$, $\nu_2 = 40$, the upper $0{\cdot}5\,\%$ point is seen to be

$$F_{0{\cdot}005}(12, 40) = 2{\cdot}9531.$$

The lower $0{\cdot}5\,\%$ point is

$$F_{0{\cdot}995}(12, 40) = 1/F_{0{\cdot}005}(40, 12) = 1/4{\cdot}2282 = 0{\cdot}2365.$$

The marginal row of each table under the heading $\nu_2 = \infty$ provides the corresponding upper percentage point of the distribution of χ^2/ν with $\nu = \nu_1$ degrees of freedom. The marginal column of each table for $\nu_1 = \infty$ gives the upper percentage point of ν/χ^2, or its reciprocal gives the lower percentage point of χ^2/ν with $\nu = \nu_2$ degrees of freedom. The calculations involved in forming these columns and rows were the basis of the tables of percentage points of χ^2 recently published in this journal (Catherine M. Thompson, 1941*b*).

The percentage points for the 'Student' ratio t having $\nu = \nu_2$ degrees of freedom may be obtained from the column of the tables headed $\nu_1 = 1$, since in this case $t = \sqrt{F}$. This relation was used to form the short table of percentage points of the t-distribution published in the last issue of this journal (Merrington, 1942).

The entries in the main body of the tables were computed by Mrs Maxine Merrington from Miss Thompson's (1941*a*) values of the percentage points of x, using the transformation

$$F = \frac{p(1-x)}{qx}.$$

The results are given to five significant figures.

In issuing the tables in present form the question of the number of figures to be retained needed some consideration. For the ordinary purpose of significance tests there is no doubt that the two-decimal place accuracy usually given for F (e.g. Snedecor (1934) and Fisher & Yates (1938)) is ample, and in a book of tables issued primarily for the working statistician this consideration would be the deciding factor. Experience, however, has shown that the table-maker can never be sure of the purposes for which the table-user may need his work, either now or at some future date. For example, Simaika (1942) has recently discussed methods of interpolating additional percentage points between tabulated pivotal values. For this purpose considerable accuracy is required for the latter values. It was therefore decided to publish these tables to the full accuracy available both in the direct (x) and inverted (F) forms of the beta distribution.

The methods of interpolation discussed by Hartley (1941) and Comrie & Hartley (1941) for the case of the x-tables will in general be applicable also for the F-tables.

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TABLES OF THE PROBABILITY INTEGRAL OF THE STUDENTIZED RANGE

By E. S. PEARSON and H. O. HARTLEY

1. Introduction

Denote by $x_1, \ldots, x_n$ a random sample of n observations arranged in ascending order of magnitude and drawn from a normal population with standard deviation σ. The range, or spread, in the sample is then $x_n - x_1$ and may be expressed in units of the standard deviation by the ratio

$$w = \frac{x_n - x_1}{\sigma}. \tag{1}$$

Tables of the probability integral of w were given in the last issue of *Biometrika* (Hartley, 1942; Pearson, 1942), together with tables of its percentage points.* The uses of the range cover a wide field. As an example of particular importance we should mention the setting up of quality control charts in industry, where the simplicity of range is of great advantage.

The probability distribution of the difference $x_n - x_1$ depends on the standard deviation σ, in the sampled population. In many cases this will be unknown, but can be estimated from a second independent sample $X_1, \ldots, X_{\nu+1}$, drawn from the same population. If we denote by

$$s^2 = \frac{1}{\nu} \sum_{1}^{\nu+1} (X - \bar{X})^2,$$

the estimate of σ^2 derived therefrom, it is easy to see that the probability law of the ratio

$$q = \frac{x_n - x_1}{s} \tag{2}$$

does not depend on σ.

This idea of 'studentizing' the range seems to have occurred first to W. S. Gosset himself (see letter of 29 January 1932 quoted by E. S. Pearson, 1938, p. 245). Following this suggestion, D. Newman (1939) calculated a number of percentage points for q. Newman's table was obtained by quadrature from E. S. Pearson's approximate probability law of w. Since exact tables of this latter integral are now available, it appeared appropriate to revise and amplify the 'studentized' distribution law resulting from it. Moreover, certain new results (to be published separately) make it possible to simplify both the calculation of 'studentized' probability laws as well as their tabulations. It suffices here to *state* these results as applied to the range.

2. Description of the tables

The probability law of q depends both on the size n of the first sample (from which the range $x_n - x_1$ is determined) and on the degrees of freedom ν of the standard deviation estimated from the second sample. The probability integral may be denoted by ${}_{\nu}P_n(Q)$ and represents the chance that the ratio q does not exceed the limit Q. As $\nu \to \infty$ and $s^2 \to \sigma^2$, ${}_{\nu}P_n(Q)$ will tend to the probability integral $P_n(W)$ of the ratio $w = (x_n - x_1)/\sigma$, taken at $W = Q$.

* See also Simon (1941, pp. 204–7). There are slight, though practically unimportant, inaccuracies in his Table C 2.

It can now be proved that to an accuracy sufficient for practical purposes and for $\nu \geqslant 10$, ${}_{\nu}P_n(Q)$ may be represented as a quadratic in $1/\nu$, viz.

$$ {}_{\nu}P_n(Q) = P_n(Q) + \frac{1}{\nu} a_n(Q) + \frac{1}{\nu^2} b_n(Q). \tag{3} $$

The coefficients $P_n(Q)$, $a_n(Q)$ and $b_n(Q)$ are given in Table 1, and the following example illustrates their use.

Example. Consider the range in samples of 12 and an independent estimate of the standard deviation based on 15 degrees of freedom. Find the chance that in random sampling the former exceeds 4 times the latter.

Entering Table 1 for $n = 12$ and $Q = 4$, we find

$$ P_{12}(4) = 0{\cdot}8321, \quad a_{12}(4) = -1{\cdot}71, \quad b_{12}(4) = 6{\cdot}2, $$

whence
$$ {}_{15}P_{12}(4) = 0{\cdot}8321 - \frac{1{\cdot}71}{15} + \frac{6{\cdot}2}{(15)^2} = 0{\cdot}8321 - 0{\cdot}114 + 0{\cdot}028 = 0{\cdot}746, $$

so that the required chance is equal to

$$ 1 - {}_{15}P_{12}(4) = 0{\cdot}254. $$

If the desired value of Q is not a tabular value, ${}_{\nu}P_n(Q)$ is found by interpolation Q-wise.

For certain applications it is necessary to know the limits Q corresponding to standard probability levels. Table 2 gives these percentage points for four levels, viz. the upper and lower 5 and 1 % points.*

The lower percentage points are given to two places of decimals, for degrees of freedom ν, ranging from 10 to ∞ and, for sample sizes n, between 2 and 20. The upper percentage points are given for the same values of ν and n and to the same accuracy, except that for $10 \leqslant \nu < 20$ only one place of decimals is given. This drop in accuracy is necessary as formula (3) is less accurate for small values of ν and large values of Q. One decimal accuracy in the percentage points is, however, considered ample having regard to the inaccuracies, which may be considerable for large Q, introduced through possible deviations from normality in the parental distribution. When using these tables it is advisable, therefore, to judge percentage points whose values exceed 6 with discretion.

A comparison of the upper percentage points given in Table 2 with those given in Newman's table (1939, p. 25) shows that while the latter may be in error by as much as 1 or 2 units in the first decimal, it provides a useful working guide to the significance levels of q.

3. Applications

We confine ourselves to a selected number of applications without claiming to cover the whole field. The first example is a modification of one used by 'Student' (1927, pp. 161–2).

Example 1. *Control of accuracy in chemical routine analysis.* The problem which 'Student' considered was the common one with which the industrial chemist is faced of making day after day a certain number of similar routine analyses of some solution or substance that must be regularly checked for conformity to standard. The characteristic measured, for example, the acidity of a solution, is estimated from the mean of a few (say n) observations, and a routine check on the consistency of these determinations is required to ensure that

* These levels are most commonly used in tests of significance, but other levels may be required. For instance, the 2·5% point and the 0·1% point are frequently used as 'inner' and 'outer' limits on quality control charts. Such additional percentage points can, of course, be calculated from Table 1 by inverse interpolation.

accuracy is maintained. Discordant observations will be repeated and, if necessary, rejected, and 'Student' pointed out that it would obviously be of advantage to work on a regular system; for this purpose he proposed the use of the range of the n determinations. The situation he considered was one in which the standard deviation of the within-day error of analysis had been found from experience to remain constant and could be assigned a known value, σ.

It is clear, however, that situations will occur in which the standard error of analysis appropriate on a given day can only be estimated from the determinations of a few previous days, perhaps because a new chemist has been put on to the work or because the method of analysis has changed. Instead, therefore, of basing the regular check system on the values of $w = (x_n - x_1)/\sigma$, as 'Student' did, we shall suppose it to be based on $q = (x_n - x_1)/s$; we then have the following modification of his example.

On each day four determinations are made in the first instance; if these show too great a scatter the data have to be improved by additional tests. Suppose that it is considered advisable to base s, the estimate of the appropriate σ, on the tests of five previous days only, i.e. on 20 observations, then ν will equal 15. In the units dealt with, let us suppose we obtain $s = 0{\cdot}675$. Then the 5 % points of Table 2 (for 15 degrees of freedom) provide gauge values for the actual range $x_n - x_1$, as follows:

Size of sample n	Gauge value $Q_n s$ for the actual range $R_n = x_n - x_1$
4	$0{\cdot}675 \times 4{\cdot}1 = 2{\cdot}8 = Q_4 s$
5	$\times 4{\cdot}4 = 3{\cdot}0 = Q_5 s$
6	$\times 4{\cdot}6 = 3{\cdot}1 = Q_6 s$
7	$\times 4{\cdot}8 = 3{\cdot}2 = Q_7 s$

'Student's' procedure is now as follows. If the actual range of a day's sample of four test results (R_4) is greater than $Q_4 s$, repetition of the test should be made giving rise to a sample of 5 with range R_5. If R_5 is smaller than $Q_5 s$ the mean of the five results should be accepted as the day's mean. If, on the other hand, R_5 exceeds $Q_5 s$ the most outlying observation should be rejected and if the resulting sample of 4 has a range R_4 smaller than $Q_4 s$ the mean of these four tests should be accepted; but if not, a further repetition should be made and the whole sample of six tests examined afresh, and so on until a sample of at least four tests with R_n smaller than $Q_n s$ is obtained.

Suppose, for example, that for a particular day the four test results are 22·8, 23·5, 26·0 and 26·6. We find that the range R_4 is 3·8 and exceeds $Q_4 s$. We therefore repeat the test, obtaining (say) 23·9 as the fifth result. For our sample of 5, $R_5 = 3{\cdot}8$, which exceeds $Q_5 s$, and we reject the result 22·8. This leaves us a sample of 4 with $R_4 = 3{\cdot}1$, which is still in excess of $Q_4 s$. We therefore repeat again, getting (say) 23·5. Now we have a sample of 6 with a range of $R_6 = 3{\cdot}8$, which exceeds $Q_6 s$. We therefore reject 22·8 again, to reach a sample of 5 with $R_5 = 3{\cdot}1$, and rejecting 26·6 a sample of 4 with $R_4 = 2{\cdot}5$. This is smaller than $Q_4 s$ and therefore we accept the mean (24·2) of the remaining test results (23·5, 26·0, 23·9, 23·5) as the day's mean.

In this particular example it happens that there is little alteration in the procedure when compared with 'Student's' procedure, which was based on the percentage points of w. In general it may be said that if there is sufficient information to show that σ remains constant from day to day, it is preferable to use a long-term estimate based on many degrees of freedom, so that the percentage points of q assume the limiting values of those for w, i.e. the values in the bottom row of Table 2 with $\nu = \infty$. To take account only of the last few

days' results in estimating σ would lead to unnecessary latitude in the gauge we are using. If, however, there are reasons for supposing that σ may change, then we must use the short-term estimate and the resulting greater uncertainty requires the wider q-limits.

In conclusion we should quote 'Student': 'It should always be remembered that such rules are to be regarded as aids to and not as substitutes for common sense!'

Example 2. *Control charts for range.* The preceding example was concerned with controlling the accuracy of a testing technique. In the usual quality control problem, charts for variability are usually concerned with real fluctuations in the quality of the manufactured article. Here the use of range in the place of standard deviation or mean deviation has the advantage of simplicity. For a full description of applications the reader is referred to Dudding & Jennett (1942), Pearson (1935) or Simon (1941).

The variability in the quality of manufactured products will often be stable and may then be adequately represented by a fixed known standard deviation σ (which in some cases is fixed by specification). With some processes, however, variability in quality is influenced by short-term factors. Although such fluctuation may be well within the permissible tolerance limits, the standard deviation will have to be estimated from a limited number of immediately preceding observations, i.e. the control limits must be based on s rather than σ and latitude must be allowed for consequent greater uncertainty. Here the percentage points for q rather than w are appropriate.

As an example consider the manufacture, on a quantity basis, of a component of an electro-mechanical instrument which has to be machined to fairly close tolerances. Suppose that the component is produced by a battery of eight machines and that during each shift samples of five components are measured (to precision) for each of the eight machines. A control chart for range is to be used to control the variability in general and the uniformity of the average performance of the eight machines in particular.

Slow secular changes in the standard deviation of the measurements are observed and a fresh estimate of the standard deviation is, therefore, calculated weekly from the first batch of eight samples of five measurements. Suppose that the standard deviation, s, within samples (based on 32 degrees of freedom) is 0·0015 inch; using now the upper and lower percentage points, we obtain by interpolation the values corresponding to $n = 5$ and $\nu = 32$ and find for the 1 % control limits:

$$\text{Upper limit} = 5{\cdot}03 \times 0{\cdot}0015 = 0{\cdot}0075 \text{ inch},$$
$$\text{Lower limit} = 0{\cdot}66 \times 0{\cdot}0015 = 0{\cdot}0010 \text{ inch}.$$

It is now desired to control the uniformity in average performance of the eight machines. To this end we may calculate for each machine the average of the five measurements* in each sample tested and plot, on a second chart, the range of the eight 'machine averages'. The control limits will be at

$$\text{Upper limit} = 5{\cdot}51 \times \frac{0{\cdot}0015}{\sqrt{5}} = 0{\cdot}0037 \text{ inch},$$

$$\text{Lower limit} = 1{\cdot}17 \times \frac{0{\cdot}0015}{\sqrt{5}} = 0{\cdot}0008 \text{ inch}.$$

Example 3. *A special problem of 'spread' in machine part assembly.* In the above examples the range was used as a simple short-cut measure of the variability of test results. It was chosen because of its simplicity rather than on theoretical grounds. There are, however,

* These averages have to be computed in any case, as they are required for the control chart for mean.

problems where the requirements of the applications (usually dictated by certain tolerance limits) demand a control of range. Instances such as the spread of salvos from a battery of guns or the spread in a stick of bombs cannot be dealt with here. We shall discuss, however, a rather special example taken from the assembly of electromagnetic machine parts and stated in a simplified form.

Electromagnetic relays are manufactured to a specified setting-up time. This is the time elapsing between the primary impulse in the coil and the complete contact in the secondary circuit of the relay. For each individual relay the actual setting-up time may differ from specification but will stay practically constant in time. Sets of (say) 15 relays are now assembled in a machine. To prevent 'arcing' a cam must break the secondary circuit before the first relay has set up until after the last relay has set up. The length of the break interval is a fixed characteristic of the cam and samples of 15 relays whose range in setting-up times exceeds this interval cannot, therefore, be fitted into the machine. Thus, on testing, the slowest or the fastest relay will have to be replaced and it is necessary to keep the frequency of such replacements below a reasonably low percentage.

Table 1 gives this frequency. Suppose, for example, in the first place that we have enough information to regard the standard deviation of the setting-up times as known and equal to 1/25th of a second; further, that the cam has a break-interval of 1/5th of a second. Then $Q = 0{\cdot}2/0{\cdot}04 = 5, n = 15$ and we find from the table that $P_{15}(5) = 0{\cdot}9688$. Thus we see that the expected frequency for replacements is

$$100(1 - P_{15}(5)) = 3{\cdot}12\ \%.$$

In general the formulae:

$$Q = \frac{\text{Cam-break-interval}}{\text{Standard deviation of setting-up times}}$$

$$100(1 - P_n(Q)) = \text{Percentage frequency of necessary replacements,}$$

$$n = \text{Number of relays assembled in the machine,}$$

relate the accuracy of the relays to the cam-break-interval. The table may, therefore, be used as a guide when deciding on tolerance limits in the manufacture of relays or in designing a cam to 'make it fit' the relays.

If the standard deviation of the setting-up times is not known exactly, but has to be estimated from a limited sample, the integral $P_n(Q)$ has to be replaced by ${}_vP_n(Q)$, which is given by formula (3).

Further applications of the tables to the Analysis of Variance of field experiments are given in Newman's (1939) paper.

4. CALCULATION OF TABLES

Both tables are based on a five-decimal manuscript table of the probability integral of range $P_n(W)$, four decimals of which were published in the last issue of *Biometrika* (Pearson, 1942). Of the coefficients given in Table 1, $P_n(Q)$ was copied from the published table, putting $Q = W$. The formulae for the coefficients a_n and b_n are

$$a_n(Q) = \frac{1}{4}\left\{Q^2\frac{d^2P_n}{dW^2} - Q\frac{dP_n}{dW}\right\},$$

$$b_n(Q) = \frac{1}{16}\left\{\frac{Q^4}{2}\frac{d^4P_n}{dW^4} - \frac{Q^3}{3}\frac{d^3P_n}{dW^3} - \frac{Q^2}{2}\frac{d^2P_n}{dW^2} + Q\frac{dP_n}{dW}\right\}.$$

The derivatives in these formulae were calculated from the differences of $P_n(W)$ (at interval 0·25) by standard formulae and are taken at argument $W = Q$. Checks consisted in differencing Q-wise and n-wise, but owing to rounding-off errors in the differences the last figure of $b_n(Q)$ cannot be guaranteed near the bottom of Table 1. A special marginal check was obtained from 'Student's' table of the probability integral of t (1925), using the relation $\sqrt{2} \times t = q$ for $n = 2$.

Table 2 was produced from Table 1 by inverse interpolation using the relation (3).

We wish to acknowledge the careful work of Mrs M. Merrington, who carried out most of the calculations.

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Table 1. *For calculating the probability integral of* $q = \frac{x_n - x_1}{s}$

	P_n	a_n	b_n	P_n	a_n	b_n	P_n	a_n	b_n
n / Q		3			4			5	
0·00	0·0000			0·0000			0·0000		
0·25	·0171			·0020			·0002		
0·50	·0666			·0152	+0·01		·0033	+0·01	
0·75	·1436	−0·02		·0483	+0·02		·0157	+0·02	
1·00	0·2407	−0·06		0·1057	+0·03		0·0450	+0·05	−0·1
1·25	·3495	−0·12		·1868	0·00		·0970	+0·07	−0·1
1·50	·4614	−0·21	+0·1	·2865	−0·06		·1733	+0·06	−0·1
1·75	·5690	−0·31	+0·2	·3970	−0·17	+0·1	·2706	−0·02	−0·1
2·00	0·6665	−0·40	+0·3	0·5096	−0·32	+0·4	0·3816	−0·16	+0·2
2·25	·7505	−0·48	+0·4	·6163	−0·47	+0·6	·4968	−0·36	+0·6
2·50	·8195	−0·53	+0·4	·7110	−0·61	+0·9	·6075	−0·57	+1·1
2·75	·8737	−0·54	+0·3	·7905	−0·69	+1·0	·7063	−0·74	+1·5
3·00	0·9145	−0·51	+0·1	0·8537	−0·72	+0·8	0·7891	−0·85	+1·6
3·25	·9439	−0·46	−0·2	·9016	−0·70	+0·5	·8546	−0·88	+1·3
3·50	·9644	−0·39	−0·4	·9361	−0·69	−0·1	·9037	−0·84	+0·7
3·75	·9782	−0·31	−0·7	·9600	−0·52	−0·6	·9386	−0·74	−0·2
4·00	0·9870	−0·24	−0·9	0·9758	−0·42	−1·1	0·9623	−0·61	−0·9
4·25	·9925	−0·18	−1·0	·9859	−0·31	−1·4	·9777	−0·47	−1·6
4·50	·9958	−0·12	−1·0	·9920	−0·22	−1·6	·9873	−0·34	−1·9
4·75	·9977	−0·08	−0·9	·9956	−0·15	−1·5	·9930	−0·24	−1·9
5·00	0·9988	−0·05	−0·7	0·9977	−0·10	−1·3	0·9963	−0·16	−1·8
5·25	·9994	−0·03	−0·5	·9988	−0·06	−1·0	·9981	−0·10	−1·6
5·50	·9997	−0·02	−0·4	·9994	−0·04	−0·8	·9990	−0·06	−1·2
5·75	·9999	−0·01	−0·3	·9997	−0·02	−0·6	·9995	−0·03	−0·9
6·00	0·9999	0·00	−0·2	0·9999	−0·01	−0·4	0·9998	−0·02	−0·6
n / Q		6			7			8	
0·50	0·0007			0·0002			0·0000		
0·75	·0050	+0·02		·0016	+0·01		·0005		
1·00	0·0188	+0·05		0·0078	+0·04		0·0032	+0·02	
1·25	·0495	+0·09	−0·1	·0250	+0·08	−0·1	·0124	+0·07	
1·50	·1031	+0·13	−0·2	·0606	+0·15	−0·2	·0353	+0·14	−0·2
1·75	·1815	+0·11	−0·3	·1204	+0·19	−0·4	·0792	+0·22	−0·5
2·00	0·2816	+0·01	−0·2	0·2056	+0·15	−0·5	0·1489	+0·25	−0·7
2·25	·3955	−0·19	+0·2	·3118	−0·01	−0·3	·2440	+0·15	−0·7
2·50	·5132	−0·44	+1·0	·4300	−0·27	+0·6	·3579	−0·09	0·0
2·75	·6252	−0·70	+1·7	·5494	−0·59	+1·6	·4800	−0·44	+1·2
3·00	0·7239	−0·90	+2·3	0·6601	−0·88	+2·7	0·5991	−0·81	+2·8
3·25	·8053	−1·02	+2·2	·7553	−1·09	+3·1	·7055	−1·10	+3·7
3·50	·8685	−1·02	+1·6	·8316	−1·16	+2·7	·7938	−1·26	+3·9
3·75	·9148	−0·94	+0·7	·8891	−1·12	+1·6	·8622	−1·28	+2·8
4·00	0·9469	−0·80	−0·6	0·9300	−0·99	+0·1	0·9120	−1·16	+1·0
4·25	·9682	−0·63	−1·5	·9576	−0·81	−1·3	·9461	−0·98	−0·7
4·50	·9817	−0·47	−2·1	·9754	−0·61	−2·4	·9684	−0·76	−2·2
4·75	·9898	−0·33	−2·4	·9862	−0·44	−3·0	·9822	−0·55	−3·4
5·00	0·9946	−0·22	−2·5	0·9926	−0·30	−3·0	0·9903	−0·38	−3·4
5·25	·9972	−0·14	−2·1	·9961	−0·19	−2·8	·9949	−0·25	−3·4
5·50	·9986	−0·09	−1·7	·9981	−0·12	−2·2	·9974	−0·15	−2·9
5·75	·9993	−0·05	−1·3	·9991	−0·07	−1·7	·9988	−0·09	−2·2
6·00	0·9997	−0·03	−0·9	0·9996	−0·04	−1·2	0·9994	−0·05	−1·5

$$_{\nu}P_n(Q) = P_n(Q) + \frac{1}{\nu} a_n(Q) + \frac{1}{\nu^2} b_n(Q)$$

Table 1 (cont.). *For calculating the probability integral of* $q=\frac{x_n-x_1}{s}$

	P_n	a_n	b_n	P_n	a_n	b_n	P_n	a_n	b_n
Q \ n		**9**			**10**			**11**	
0·75	0·0002			0·0000			0·0000		
1·00	0·0013	+0·01		0·0005	+0·01		0·0002		
1·25	·0062	+0·05		·0030	+0·03		·0015	+0·02	+0·1
1·50	·0204	+0·12	−0·1	·0117	+0·10		·0067	+0·07	+0·1
1·75	·0517	+0·23	−0·4	·0336	+0·21	−0·3	·0217	+0·18	−0·1
2·00	0·1072	+0·30	−0·8	0·0768	+0·33	−0·8	0·0548	+0·33	−0·7
2·25	·1899	+0·28	−1·1	·1470	+0·37	−1·4	·1134	+0·43	−1·5
2·50	·2964	+0·08	−0·6	·2443	+0·24	−1·3	·2007	+0·37	−1·8
2·75	·4175	−0·26	+0·6	·3617	−0·08	−0·2	·3124	+0·10	−1·0
3·00	0·5415	−0·68	+2·6	0·4878	−0·53	+2·1	0·4382	−0·36	+1·4
3·25	·6569	−1·06	+4·2	·6099	−0·99	+4·3	·5649	−0·88	+4·1
3·50	·7558	−1·32	+4·9	·7180	−1·34	+5·6	·6807	−1·32	+6·3
3·75	·8345	−1·40	+4·0	·8062	−1·50	+5·4	·7776	−1·57	+6·6
4·00	0·8929	−1·33	+2·2	0·8731	−1·47	+3·4	0·8528	−1·60	+4·8
4·25	·9338	−1·14	+0·1	·9208	−1·31	+1·0	·9072	−1·46	+2·2
4·50	·9608	−0·91	−2·1	·9527	−1·06	−1·8	·9441	−1·20	−1·2
4·75	·9777	−0·67	−3·6	·9729	−0·80	−3·5	·9678	−0·92	−3·3
5·00	0·9878	−0·47	−4·0	0·9851	−0·56	−4·6	0·9822	−0·65	−5·0
5·25	·9936	−0·31	−3·9	·9922	−0·37	−4·4	·9906	−0·44	−5·0
5·50	·9968	−0·19	−3·5	·9960	−0·23	−4·1	·9952	−0·28	−4·6
5·75	·9984	−0·11	−2·7	·9980	−0·14	−3·3	·9976	−0·16	−3·8
6·00	0·9993	−0·07	−1·8	0·9991	−0·08	−2·3	0·9989	−0·09	−2·8
6·25	·9997	−0·03	−1·3	·9996	−0·04	−1·7	·9995	−0·05	−1·9
6·50	·9998	−0·02	−0·8	·9998	−0·02	−0·9	·9998	−0·03	−1·1
Q \ n		**12**			**13**			**14**	
1·00	0·0001			0·0000			0·0000		
1·25	·0007	+0·01	+0·1	·0004	+0·01		·0002		
1·50	·0038	+0·05	+0·1	·0022	+0·04	+0·1	·0012	+0·03	+0·1
1·75	·0140	+0·15	0·0	·0090	+0·12	+0·1	·0058	+0·10	+0·2
2·00	0·0389	+0·31	−0·5	0·0276	+0·28	−0·3	0·0195	+0·24	−0·1
2·25	·0872	+0·46	−1·5	·0669	+0·46	−1·4	·0511	+0·45	−1·1
2·50	·1644	+0·47	−2·2	·1342	+0·55	−2·4	·1094	+0·59	−2·5
2·75	·2690	+0·27	−1·8	·2311	+0·41	−2·5	·1981	+0·54	−3·3
3·00	0·3927	−0·18	+0·5	0·3512	0·0	−0·5	0·3134	+0·17	−1·6
3·25	·5222	−0·74	+3·7	·4817	−0·59	+3·0	·4437	−0·42	+2·1
3·50	·6442	−1·26	+6·7	·6087	−1·18	+6·8	·5744	−1·08	+6·7
3·75	·7491	−1·61	+7·6	·7206	−1·61	+8·5	·6925	−1·60	+9·2
4·00	0·8321	−1·71	+6·2	0·8111	−1·80	+7·7	0·7899	−1·86	+9·1
4·25	·8931	−1·60	+3·4	·8787	−1·73	+4·7	·8639	−1·85	+6·2
4·50	·9352	−1·35	−0·5	·9258	−1·49	+0·5	·9162	−1·63	+1·6
4·75	·9624	−1·05	−3·1	·9567	−1·18	−2·7	·9508	−1·30	−2·2
5·00	0·9791	−0·75	−5·2	0·9759	−0·85	−5·3	0·9724	−0·96	−5·3
5·25	·9889	−0·51	−5·6	·9871	−0·58	−6·0	·9852	−0·66	−6·3
5·50	·9943	−0·32	−5·0	·9934	−0·37	−5·6	·9924	−0·42	−6·3
5·75	·9972	−0·19	−4·3	·9967	−0·22	−4·8	·9962	−0·26	−5·4
6·00	0·9987	−0·11	−3·1	0·9984	−0·13	−3·7	0·9982	−0·15	−4·2
6·25	·9994	−0·06	−2·2	·9993	−0·07	−2·6	·9992	−0·08	−2·9
6·50	·9997	−0·03	−1·5	·9997	−0·03	−1·6	·9996	−0·04	−1·8

$$_{\nu}P_n(Q)=P_n(Q)+\frac{1}{\nu}a_n(Q)+\frac{1}{\nu^2}b_n(Q)$$

Table 1 (cont.). *For calculating the probability integral of* $q = \dfrac{x_n - x_1}{s}$

Q \ n	P_n	a_n	b_n	P_n	a_n	b_n	P_n	a_n	b_n
		15			**16**			**17**	
1·00	0·0000			0·0000			0·0000		
1·25	·0001			·0000			·0000		
1·50	·0007	+0·02	+ 0·1	·0004	+0·01	+ 0·1	·0002	+0·01	+ 0·1
1·75	·0037	+0·07	+ 0·2	·0024	+0·06	+ 0·2	·0015	+0·04	+ 0·2
2·00	0·0137	+0·21	0·0	0·0097	+0·18	+ 0·2	0·0068	+0·14	+ 0·3
2·25	·0390	+0·42	− 0·9	·0297	+0·39	− 0·6	·0226	+0·35	− 0·4
2·50	·0890	+0·62	− 2·5	·0722	+0·62	− 2·3	·0586	+0·62	− 2·0
2·75	·1696	+0·64	− 3·8	·1448	+0·72	− 4·1	·1236	+0·77	− 4·3
3·00	0·2792	+0·34	− 2·6	0·2484	+0·49	− 3·6	0·2207	+0·62	− 4·5
3·25	·4081	−0·25	+ 1·0	·3748	−0·07	− 0·2	·3438	+0·10	− 1·4
3·50	·5413	−0·96	+ 6·2	·5096	−0·82	+ 5·5	·4792	−0·67	+ 4·7
3·75	·6648	−1·56	+ 9·8	·6376	−1·50	+10·1	·6110	−1·42	+10·1
4·00	0·7686	−1·91	+10·4	0·7474	−1·94	+11·6	0·7263	−1·95	+12·8
4·25	·8488	−1·96	+ 7·7	·8336	−2·05	+ 9·2	·8182	−2·13	+10·7
4·50	·9062	−1·76	+ 2·8	·8960	−1·88	+ 4·1	·8856	−2·00	+ 5·5
4·75	·9446	−1·43	− 1·6	·9383	−1·55	− 0·8	·9317	−1·68	+ 0·2
5·00	0·9688	−1·06	− 5·3	0·9650	−1·17	− 5·2	0·9611	−1·28	− 5·0
5·25	·9832	−0·73	− 6·8	·9811	−0·82	− 7·1	·9789	−0·90	− 7·3
5·50	·9913	−0·48	− 6·8	·9902	−0·53	− 7·5	·9890	−0·59	− 8·1
5·75	·9957	−0·29	− 5·8	·9951	−0·33	− 6·4	·9945	−0·37	− 7·1
6·00	0·9979	−0·17	− 4·7	0·9976	−0·19	− 5·3	0·9974	−0·21	− 5·8
6·25	·9990	−0·09	− 3·4	·9989	−0·11	− 3·7	·9988	−0·12	− 4·1
6·50	·9996	−0·05	− 2·1	·9995	−0·05	− 2·3	·9995	−0·06	− 2·6
Q \ n		**18**			**19**			**20**	
1·00	0·0000			0·0000			0·0000		
1·25	·0000			·0000			·0000		
1·50	·0001		+ 0·1	·0001		+ 0·1	·0000		
1·75	·0010	+0·03	+ 0·2	·0006	+0·02	+ 0·2	·0004	+0·02	+ 0·2
2·00	0·0048	+0·12	+ 0·4	0·0033	+0·10	+ 0·4	0·0023	+0·08	+ 0·4
2·25	·0172	+0·32	− 0·1	·0130	+0·28	+ 0·2	·0099	+0·24	+ 0·4
2·50	·0474	+0·59	− 1·7	·0383	+0·56	− 1·4	·0309	+0·53	− 1·0
2·75	·1053	+0·81	− 4·4	·0896	+0·83	− 4·3	·0761	+0·83	− 4·0
3·00	0·1959	+0·73	− 5·2	0·1736	+0·83	− 5·9	0·1538	+0·91	− 6·4
3·25	·3151	+0·27	− 2·7	·2884	+0·42	− 4·0	·2638	+0·57	− 5·2
3·50	·4502	−0·51	+ 3·6	·4226	−0·35	+ 2·4	·3964	−0·18	+ 1·1
3·75	·5850	−1·33	+ 9·9	·5598	−1·22	+ 9·6	·5352	−1·10	+ 8·9
4·00	0·7053	−1·94	+13·7	0·6845	−1·92	+14·5	0·6640	−1·88	+15·2
4·25	·8027	−2·19	+12·2	·7871	−2·25	+13·7	·7715	−2·29	+15·1
4·50	·8750	−2·11	+ 7·1	·8643	−2·21	+ 8·7	·8534	−2·30	+10·2
4·75	·9249	−1·80	+ 1·2	·9180	−1·91	+ 2·3	·9110	−2·03	+ 3·6
5·00	0·9571	−1·38	− 4·7	0·9529	−1·49	− 4·2	0·9486	−1·59	− 3·7
5·25	·9766	−0·98	− 7·4	·9742	−1·07	− 7·4	·9718	−1·16	− 7·3
5·50	·9878	−0·65	− 8·5	·9865	−0·71	− 8·7	·9852	−0·77	− 8·9
5·75	·9939	−0·40	− 7·9	·9932	−0·44	− 8·4	·9925	−0·48	− 8·8
6·00	0·9971	−0·24	− 6·3	0·9967	−0·26	− 6·9	0·9964	−0·29	− 7·4
6·25	·9986	−0·13	− 4·6	·9985	−0·14	− 5·1	·9983	−0·16	− 5·5
6·50	·9994	−0·07	− 3·0	·9993	−0·08	− 3·3	·9992	−0·08	− 3·6

$$_{p}P_n(Q) = P_n(Q) + \frac{1}{p}\,a_n(Q) + \frac{1}{p^2}\,b_n(Q)$$

Table 2. *Lower percentage points of the studentized range* $q = \dfrac{x_n - x_1}{s}$.

5% points

ν \ n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
10	0·09	0·43	0·75	1·01	1·20	1·37	1·52	1·63	1·74	1·83	1·91	1·98	2·05	2·12	2·17	2·22	2·26	2·30	2·34
11	·09	·43	·75	·01	·21	·38	·52	·64	·75	·84	·92	2·00	·07	·13	·18	·24	·28	·33	·37
12	·09	·43	·75	·01	·21	·38	·53	·65	·76	·85	·93	·01	·08	·14	·20	·26	·30	·34	·38
13	·09	·43	·75	·01	·22	·39	·53	·65	·76	·86	·94	·02	·09	·15	·21	·27	·31	·36	·40
14	·09	·43	·75	·01	·22	·39	·54	·66	·77	·86	·95	·03	·10	·16	·22	·28	·32	·37	·41
15	0·09	0·43	0·75	1·01	1·22	1·39	1·54	1·66	1·77	1·87	1·95	2·03	2·11	2·17	2·23	2·29	2·34	2·38	2·43
16	·09	·43	·75	·01	·22	·39	·54	·67	·78	·87	·96	·04	·11	·18	·24	·30	·34	·39	·44
17	·09	·43	·75	·01	·22	·40	·55	·67	·78	·88	·97	·05	·12	·19	·25	·30	·35	·40	·45
18	·09	·43	·75	·02	·22	·40	·55	·67	·79	·88	·97	·05	·12	·19	·25	·31	·36	·41	·45
19	·09	·43	·75	·02	·23	·40	·55	·68	·79	·89	·98	·05	·13	·20	·26	·32	·37	·42	·46
20	0·09	0·43	0·75	1·02	1·23	1·40	1·55	1·68	1·79	1·89	1·98	2·06	2·13	2·20	2·27	2·32	2·37	2·42	2·47
24	·09	·43	·75	·02	·23	·41	·56	·69	·80	·90	·99	·08	·15	·22	·28	·34	·39	·45	·49
30	·09	·43	·76	·02	·24	·41	·57	·70	·81	·92	2·01	·09	·17	·24	·30	·36	·41	·47	·52
40	·09	·43	·76	·02	·24	·42	·57	·71	·82	·93	·02	·10	·18	·26	·32	·38	·43	·49	·54
60	0·09	0·43	0·76	1·02	1·24	1·43	1·58	1·72	1·83	1·94	2·04	2·12	2·20	2·28	2·34	2·40	2·46	2·52	2·57
120	·09	·43	·76	·03	·25	·43	·59	·73	·85	·96	·06	·14	·22	·30	·36	·43	·49	·54	·60
∞	0·09	0·43	0·76	1·03	1·25	1·44	1·60	1·74	1·86	1·97	2·07	2·16	2·24	2·32	2·39	2·45	2·52	2·57	2·62

1% points

ν \ n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
10	0·02	0·18	0·42	0·64	0·81	0·96	1·11	1·23	1·34	1·41	1·50	1·57	1·62	1·70	1·74	1·81	1·84	1·88	1·92
11	·02	·18	·42	·64	·82	·97	·12	·24	·35	·43	·52	·58	·64	·71	·76	·82	·86	·91	·94
12	·02	·18	·42	·64	·82	·98	·12	·24	·35	·44	·53	·60	·65	·73	·77	·84	·88	·92	·96
13	·02	·18	·42	·64	·83	·98	·13	·25	·36	·45	·54	·61	·66	·74	·79	·85	·89	·94	·98
14	·02	·18	·42	·65	·83	·99	·13	·25	·37	·46	·55	·62	·68	·76	·80	·87	·91	·95	2·00
15	0·02	0·18	0·42	0·65	0·83	0·99	1·14	1·26	1·37	1·46	1·55	1·63	1·69	1·76	1·81	1·88	1·92	1·97	2·01
16	·02	·18	·42	·65	·83	·99	·14	·26	·37	·47	·56	·63	·70	·77	·82	·89	·93	·98	·02
17	·02	·18	·42	·65	·84	1·00	·14	·27	·38	·48	·57	·64	·70	·78	·83	·90	·94	·99	·04
18	·02	·18	·42	·65	·84	·00	·15	·27	·38	·48	·57	·65	·71	·79	·84	·91	·95	2·00	·05
19	·02	·18	·43	·65	·84	·00	·15	·28	·39	·48	·58	·65	·72	·80	·85	·91	·96	·01	·06
20	0·02	0·18	0·43	0·65	0·84	1·01	1·15	1·28	1·39	1·49	1·58	1·66	1·72	1·80	1·85	1·92	1·97	2·01	2·06
24	·02	·18	·43	·65	·85	·01	·16	·29	·40	·50	·60	·67	·74	·82	·88	·94	·99	·05	·09
30	·02	·18	·43	·66	·85	·02	·17	·30	·41	·52	·61	·69	·76	·84	·90	·97	2·02	·07	·12
40	·02	·18	·43	·66	·85	·02	·18	·31	·43	·53	·63	·71	·79	·86	·92	·99	·04	·10	·15
60	0·02	0·18	0·43	0·66	0·86	1·03	1·19	1·32	1·44	1·55	1·64	1·73	1·81	1·88	1·95	2·02	2·07	2·13	2·18
120	·02	·18	·43	·66	·86	·04	·20	·33	·45	·56	·66	·75	·83	·91	·98	·04	·10	·16	·21
∞	0·02	0·19	0·43	0·66	0·87	1·05	1·20	1·34	1·47	1·58	1·68	1·77	1·86	1·93	2·01	2·08	2·14	2·20	2·25

Table 2 (cont.). *Upper percentage points of the studentized range* $q = \dfrac{x_n - x_1}{s}$.

5% points

ν \ n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
10	3·15	3·9	4·4	4·8	5·0	5·3	5·4	5·6	5·7	5·8	5·9	6·0	6·1	6·1	6·2	6·3	6·4	6·4	6·5
11	·11	·8	·3	·6	4·9	·1	·3	·5	·6	·7	8	5·9	·0	·0	·1	·2	·3	·3	·4
12	·08	·8	·2	·6	·8	·1	·2	·4	·5	·6	·7	·8	5·9	·0	·0	·1	·2	·2	·3
13	·06	·7	·2	·5	·8	·0	·1	·3	·4	·5	·6	·7	·8	5·9	·0	·0	·1	·1	·2
14	·03	·7	·1	·4	·7	4·9	·1	·2	·3	·5	·6	·7	·7	·8	5·9	·0	·0	·1	·1
15	3·01	3·7	4·1	4·4	4·6	4·8	5·0	5·2	5·3	5·4	5·5	5·6	5·7	5·7	5·8	5·9	6·0	6·0	6·1
16	·00	·7	·1	·4	·6	·8	·0	·1	·2	·3	·4	·5	·6	·7	·8	·8	5·9	·0	·0
17	2·98	·6	·0	·3	·5	·8	4·9	·1	·2	·3	·4	·5	·6	·6	·7	·8	·8	5·9	·0
18	·97	·6	·0	·3	·5	·7	·9	·0	·1	·2	·3	·4	·5	·6	·7	·7	·8	·9	5·9
19	·96	·6	·0	·3	·5	·7	·8	·0	·1	·2	·3	·4	·5	·5	·6	·7	·7	·8	·8
20	2·95	3·58	3·97	4·25	4·46	4·65	4·80	4·94	5·06	5·16	5·25	5·34	5·42	5·50	5·57	5·63	5·69	5·75	5·80
24	·92	·53	·90	·18	·38	·55	·70	·83	4·95	·05	·14	·22	·29	·36	·43	·49	·55	·60	·65
30	·89	·49	·85	·11	·30	·47	·61	·73	·84	4·93	·02	·10	·17	·23	·30	·36	·41	·46	·50
40	·86	·44	·79	·04	·23	·39	·52	·64	·74	·83	4·91	4·98	·05	·11	·17	·23	·28	·33	·37
60	2·83	3·40	3·74	3·98	4·16	4·32	4·44	4·55	4·65	4·73	4·81	4·88	4·94	5·00	5·06	5·11	5·16	5·20	5·24
120	·80	·36	·69	·92	·10	·24	·36	·47	·56	·64	·71	·78	·84	4·90	4·95	·00	·04	·09	·13
∞	2·77	3·32	3·63	3·86	4·03	4·17	4·29	4·39	4·47	4·55	4·62	4·68	4·74	4·80	4·84	4·89	4·93	4·97	5·01

1% points

ν \ n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
10	4·48	5·1	5·6	5·9	6·1	6·3	6·4	6·5	6·6	6·7	6·8	6·8	6·9	6·9	7·0	7·0	7·0	7·1	7·1
11	·39	·0	·5	·8	·0	·2	·3	·4	·5	·6	·7	·7	·8	·9	6·9	6·9	·0	·0	·1
12	·32	·0	·4	·7	5·9	·1	·2	·4	·5	·6	·6	·7	·7	·8	·9	·9	6·9	·0	·0
13	·26	4·9	·3	·6	·8	·0	·1	·3	·4	·5	·6	·6	·7	·7	·8	·8	·8	6·9	·0
14	·21	·9	·3	·6	·8	5·9	·1	·2	·4	·5	·5	·6	·6	·7	·7	·8	·8	·9	6·9
15	4·17	4·8	5·2	5·5	5·7	5·9	6·0	6·2	6·3	6·4	6·5	6·5	6·6	6·6	6·7	6·7	6·8	6·8	6·9
16	·13	·8	·2	·4	·7	·8	·0	·1	·2	·4	·4	·5	·5	·6	·6	·7	·7	·8	·8
17	·10	·7	·1	·4	·6	·8	5·9	·1	·2	·3	·3	·4	·5	·5	·6	·6	·7	·7	·8
18	·07	·7	·1	·4	·6	·7	·9	·0	·1	·2	·3	·4	·4	·5	·5	·6	·6	·7	·7
19	·05	·7	·1	·3	·5	·7	·8	·0	·1	·2	·2	·3	·4	·4	·5	·5	·6	·6	·7
20	4·02	4·65	5·02	5·30	5·51	5·67	5·80	5·93	6·04	6·12	6·20	6·27	6·34	6·40	6·45	6·51	6·56	6·61	6·66
24	3·96	·55	4·91	·17	·38	·54	·68	·80	5·90	5·98	·07	·14	·20	·26	·32	·38	·43	·48	·52
30	·89	·46	·81	·05	·24	·40	·54	·65	·76	·83	5·92	5·99	·05	·12	·19	·24	·28	·33	·37
40	·82	·37	4·69	4·94	·11	·26	·40	·51	·60	·68	·76	·83	5·90	5·96	·02	·07	·12	·16	·21
60	3·76	4·29	4·59	4·81	4·99	5·13	5·26	5·36	5·44	5·53	5·60	5·67	5·73	5·78	5·84	5·89	5·94	5·98	6·02
120	·70	·20	·50	·71	·87	·01	·12	·22	·30	·37	·44	·51	·56	·61	·66	·71	·75	·79	5·83
∞	3·64	4·12	4·40	4·60	4·76	4·88	4·99	5·08	5·16	5·23	5·29	5·35	5·40	5·45	5·49	5·53	5·57	5·61	5·64

MISCELLANEA

Minimum range for quasi-normal distributions

By R. C. GEARY

For one variate a *quasi-normal** distribution $f(x)\,dx$ is defined as follows:

(i) $f(x)$ is continuous for $-\infty \leqslant x \leqslant +\infty$;

(ii) the distribution has a single mode;

(iii) for all values of x less than the mode $f'(x)$ is non-negative, and for all values of x greater than the mode $f'(x)$ is negative.

It will first be shown that corresponding to a given probability α, the range of y, from x to $x+y$, is shortest when

$$f(x) = f(x+y). \tag{1}$$

The property is almost obvious from geometrical considerations; it may be well, however, to give an algebraic proof:

$$1-\alpha = \int_x^{x+y} f(x)\,dx. \tag{2}$$

Differentiating,

$$0 = f(x+y)\,(dx+dy) - f(x)\,dx.$$

Hence

$$\frac{dy}{dx} = \frac{f(x)}{f(x+y)} - 1, \tag{3}$$

which assumes a limiting value for $dy/dx = 0$ or

$$f(x) = f(x+y).$$

Also, from (3),

$$\frac{d^2y}{dx^2} = \{f(x+y)f'(x) - f'(x+y)f(x)\}/\{f(x+y)\}^2, \tag{4}$$

when $dy/dx = 0$. From (ii) above it is clear that the right side of (4) is positive. Hence $dy/dx = 0$ defines a minimum value of y in terms of x. From (1) and (2) the values of x and y are theoretically determinable.

This property can readily be generalized. A quasi-normal multiple frequency distribution

$$f(x_1, x_2, \ldots, x_k)\,dx_1\,dx_2 \ldots dx_k,$$

is defined as follows:

(i) $f(x_1, x_2, \ldots, x_k)$ is continuous in each of the variates from $-\infty$ to $+\infty$;

(ii) the distribution has a single mode;

(iii) each distribution in one dimension linearly derivable from $f(x_1, x_2, \ldots, x_k)$ is quasi-normal in the sense explained above for one variate, i.e. the distribution on the straight line $L_i(x_1, x_2, \ldots, x_k) = 0$, $i = 1, 2, \ldots, k-1$, where the L_i are of one dimension in the x_j, is quasi-normal. It will be shown that of all the 'surfaces' Σ which satisfy

$$1-\alpha = \int_k \cdots \int_\Sigma f(x_1, x_2, \ldots, x_k)\,dx_1\,dx_2 \ldots dx_k, \tag{5}$$

the surface of smallest 'volume' is given by

$$f(x_1, x_2, \ldots, x_k) = C, \quad \text{a constant.} \tag{6}$$

In (5) integration is extended to the inside of Σ which, like (6), is assumed to be closed.

Suppose, in fact, that at any two points $E'(x_1', x_2', \ldots, x_k')$, $E''(x_1'' x_2'', \ldots, x_k'')$ on Σ, $f(E') \neq f(E'')$, while Σ has its minimum volume. By an orthogonal change of variables x into Z the new Z_k axis can be made parallel to the line joining E' and E''. In the first stage of the integration of the right side of (5) one element will consist of

$$dZ_1\,dZ_2 \ldots dZ_{k-1} \int_{Z_k'}^{Z_k''} F(Z_1^0, Z_2^0, \ldots, Z_{k-1}^0, Z_k)\,dZ_k, \tag{7}$$

where F is the transform of f and the points E' and E'' have become respectively $(Z_1^0, Z_2^0, \ldots, Z_{k-1}^0, Z_k')$ and $(Z_1^0, Z_2^0, \ldots, Z_{k-1}^0, Z_k'')$. From (iii) and since $F(E') \neq F(E'')$ it is clear that without changing

* It is a pity that the term *normal* cannot be applied to this system (which includes the great majority of distributions met with in practice), the term *Gaussian* being reserved exclusively for the distribution generally known as 'normal' by writers in English.

$Z_1^0, Z_2^0, \ldots, Z_{k-1}^0$ or $dZ_1, dZ_2, \ldots, dZ_{k-1}$ two values of Z_k, say Y_k' and Y_k'' different from Z_k' or Z_k'', can be found, so that the value of (7) is unchanged but for which

$$|Y_k' - Y_k''| < |Z_k' - Z_k''|. \tag{8}$$

The surface Σ can then be continuously distorted to pass through the points $(Z_1^0, Z_2^0, \ldots, Z_{k-1}^0, Y_k')$ and $(Z_1^0, Z_2^0, \ldots, Z_{k-1}^0, Y_k'')$ with the obvious result that its volume will be diminished which is contrary to hypothesis. Hence at each point on Σ

$$f(x_1, x_2, \ldots, x_k) = C, \tag{9}$$

where the constant C is derivable from (5).

This property gives particular significance to the concept which Karl Pearson (1900) used in the derivation of the χ^2-test. Pearson's idea of integrating the $(k-1)$-dimensional integral

$$C \int \ldots \int_{\Sigma} e^{-\frac{1}{2}\chi^2} d\Omega = 1 - \alpha, \tag{10}$$

where

$$\chi^2 = \sum_{i=1}^{k} \frac{(x_i - \bar{x}_i)^2}{\bar{x}_i},$$

with

$$\sum_{i=1}^{k} (x_i - \bar{x}_i) = 0,$$

within the surface $\chi^2 = \text{constant}$, was simply a brilliant mathematical device for reducing the probability integral at (10) from $k-1$ to one dimension. There is an infinity of surfaces Σ which have the property indicated at (10). It will now be observed that of all the possible surfaces, that which has the smallest value of

$$\int \ldots \int_{\Sigma} d\Omega$$

in the plane $\Sigma x_i = \Sigma \bar{x}_i$ is $\chi^2 = \text{constant}$, the constant value being determined by (10).

In connexion with the estimation from random samples of parameters entering into the parent distribution, there is the analogous problem of determining the shortest fiducial or confidence interval corresponding to a probability α. In the case of one unknown parameter, Neyman (1937) has shown that statements can be made in the form

$$F_1(x_1, x_2, \ldots, x_k) \leqslant \theta \leqslant F_2(x_1, x_2, \ldots, x_k),$$

where the observed sample is $x_1, x_2, \ldots, x_k$ and where the functions F_1 and F_2 are determinable from the parent distribution function $f(x, \theta)$, in the sense that if the experiment be repeated many times the statement will be true in approximately 100 $(1-\alpha)$ % of cases and false in about 100α % of cases, α having some value like 0·05, 0·01, etc., determined in advance. If a single sufficient statistic y (a function of $x_1, x_2, \ldots, x_k$) is available in respect of θ, the statement assumes a very simple form. The limits of range of θ, namely θ_1 and θ_2, are given by the equations

$$\int_{-\infty}^{X} \phi(y, \theta_1)\, dy = \xi, \quad \int_{X}^{\infty} \phi(y, \theta_2)\, dy = \eta, \quad \xi + \eta = \alpha, \tag{11}$$

when $\phi(y, \theta)$, the distribution function of y, has the property that, corresponding to each pair of positive quantities ξ and η consistent with $\xi + \eta = \alpha$ given in advance, θ_1 and θ_2 are monotonic functions of X, the sample value of y. Theoretically θ_1 and θ_2 are determinable from (11) in terms of ξ and it would appear that the solution required is the value of ξ which minimizes $|\theta_1 - \theta_2|$. Actually the problem is more complicated because $|\theta_1 - \theta_2|$ may not be the most suitable definition of range. For example, $|\log\theta_1 - \log\theta_2|$ might be taken and, in general, the value of ξ which minimizes $|\theta_1 - \theta_2|$ will be different from that which minimizes $|\log\theta_1 - \log\theta_2|$. Neyman (1937) attempted to avoid the difficulty by adopting a probabilistic concept but he shows that this particular concept is generally inapplicable. In the following applications the metrical standard is used.

I. *Parameter of position*

Suppose that the parameter is one of position only. Equations (11) then become

$$\int_{-\infty}^{x} f(x - \theta_1)\, dx = \xi, \quad \int_{x}^{\infty} f(x - \theta_2)\, dx = \eta, \quad \xi + \eta = \alpha, \tag{12}$$

where x is the sample value of the sufficient statistic.

Set $z = \theta_1 - \theta_2$. Then, since $d\xi + d\eta = 0$,

$$\frac{dz}{d\theta_1} = 1 - \frac{f(x-\theta_1)}{f(x-\theta_2)}, \tag{13}$$

and

$$\frac{d^2z}{d\theta_1^2} = \{f(x-\theta_2)f'_x(x-\theta_1) - f(x-\theta_1)f'_x(x-\theta_2)\}/\{f(x-\theta_2)\}^2. \tag{14}$$

From (13) $dz/d\theta_1 = 0$ gives

$$f(x-\theta_1) = f(x-\theta_2), \tag{15}$$

which from (14) shows a minimum solution when $f'_x(x-\theta_1) > 0$ and $f'_x(x-\theta_2) < 0$. In the Gaussian case

$$2x = \theta_1 + \theta_2. \tag{16}$$

Equations (12) and (15) (or, in the Gaussian case, (16)) will give the values of θ_1 and θ_2 corresponding to the probability α.

II. *Estimation of m from t*

For a normal sample of n the mean of the universe m will be estimated from the Gosset-Fisher function t. The mean x and the variance s^2 having been given from the random sample of n, the requisite equations are

$$C\int_{-\infty}^{x}\left(1+\frac{t^2}{n-1}\right)^{-\frac{1}{2}n}dt = \xi, \quad C\int_{y}^{\infty}\left(1+\frac{t^2}{n-1}\right)^{-\frac{1}{2}n}dt = \eta, \quad \xi+\eta = \alpha, \tag{17}$$

with

$$x = (\bar{x}-m_1)\sqrt{n}/s, \quad y = (\bar{x}-m_2)\sqrt{n}/s.$$

Set $z = m_1 - m_2 = s(y-x)/\sqrt{n}$.
From $d\xi + d\eta = 0$ and setting $dz/dx = 0$, on reduction,

$$z = -2xs/\sqrt{n},$$

which is equivalent to

$$2\bar{x} = m_1 + m_2. \tag{18}$$

From (17) and (18), m_1 and m_2 are determinable.

III. *Estimation of the variance*

The estimation of the variance from a Gaussian sample of n illustrates the difficulty referred to above about finding a suitable standard for determining the shortest interval. In the Gaussian case the universal mean determines the position of the curve and the variance its scale, so that intuitively one feels that $|\theta_1 - \theta_2|$ is a suitable measure of range for the first and θ_1/θ_2 or $|\log\theta_1 - \log\theta_2|$ is the best measure for the second. The equations are then as follows:

$$C\int_0^v e^{-(n-1)w/2\theta_1}\left(\frac{w}{\theta_1}\right)^{\frac{1}{2}(n-3)}d\left(\frac{w}{\theta_1}\right) = \xi, \quad C\int_v^\infty e^{-(n-1)w/2\theta_2}\left(\frac{w}{\theta_2}\right)^{\frac{1}{2}(n-3)}d\left(\frac{w}{\theta_2}\right) = \eta, \quad \text{with} \quad \xi+\eta = \alpha, \tag{19}$$

where v is the variance calculated from the random sample of n. The constant C depends on n only.

Set

$$\frac{v}{\theta_1} = x, \quad \frac{v}{\theta_2} = y \quad \text{and} \quad z = \frac{y}{x} = \frac{\theta_1}{\theta_2}. \tag{20}$$

The conditon $d\xi + d\eta = 0$ gives

$$\frac{dz}{dx} = \left[\exp\left\{-\frac{n-1}{2}x(1-z)\right\} - z^{\frac{1}{2}(n-1)}\right] \Big/ xz^{\frac{1}{2}(n-3)},$$

so that, for $dz/dx = 0$,

$$x(z-1) = \log z. \tag{21}$$

To find an approximate solution of this equation set

$$z = 1 + t/\sqrt{n}, \quad x = 1 - u/\sqrt{n}.$$

Then t can be expanded as follows (to $n^{-\frac{1}{2}}$)

$$t = 2u + \tfrac{2}{3}u^3n^{-\frac{1}{2}} + \tfrac{2}{9}u^3n^{-1} + \tfrac{4}{135}u^4n^{-\frac{1}{2}} + \ldots, \tag{22}$$

or

$$z = 1 + 2(1-x) + \tfrac{2}{3}(1-x)^2 + \tfrac{2}{9}(1-x)^3 + \tfrac{4}{135}(1-x)^4 + \ldots, \tag{23}$$

correct to n^{-2}. Comparison with

$$z' = x^{-2} = (1-\overline{1-x})^{-2} = 1 + 2(1-x) + 3(1-x)^2 + 4(1-x)^3 + 5(1-x)^4 + \ldots, \tag{24}$$

shows that, to $n^{-\frac{1}{2}}$,

$$z = x^{-2},$$

or

$$v^2 = \theta_1\theta_2 \text{ approximately.} \tag{25}$$

From (19) and (23) (or approximately by (25)), θ_1 and θ_2 giving minimum θ_1/θ_2 can be determined. The analogy will be evident between (18), where the sample mean appears as the arithmetic mean of the two limits of estimate, and (25) where the sample variance is the geometric mean of the limits of estimate. The former relation is, however, absolute while the latter is only approximate.

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